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A NOTE ON ASYMMETRY SETS

In this note we give a difference between measure and category in terms of asymmetry sets. A category analogue of an approximate asymmetry set is σ-well porous, see [6]. Here, we construct a function f for which the approximate asymmetry set is not σ-well porous. In other words, the thesis that every approximate asymmetry set is σ-porous cannot be strenghthened to a thesis that every such set is σ-well porous.

Let f be a function from R into R. The asymmetry set of f is denoted by $A(f)$ and defined to be the set of all points $x \in R$ for which $W_-(f,x) \neq W_+(f,x)$ where $W_-(f,x)$, $W_+(f,x)$ denote one sided approximate cluster sets of f at a point x. More precisely, $W_+(f,x)$ is the set of all $y \in R \cup \{-\infty,+\infty\}$ satisfying the following condition, for every neighbourhood U of y, x is not a dispersion point of $f^{-1}(U)$ from the right in the sense of measure. In an analogous way is defined the set $W_-(f,x)$. As in [2] we define the category analogues of one sided dispersion as follows. Let I denote the σ-ideal of all meager sets in R. Let $B \subsetneq R$ be a Baire set.

We say that 0 is an I-dispersion point of the set B from the right if and only if for every increasing sequence of positive integers m_n there exist a subsequence m_{k_n} and $A \in I$ such that $\chi_{m_{k_n} B \cap [0,1]}(x)$ converges to 0 for all $x \in [0,1] \setminus A$. In this case we write $I\text{-}d_+(0,B) = 0$. We write $I\text{-}d_+(x,B) = 0$ if $I\text{-}d_+(0,B-x) = 0$, where $B-x = \{b-x\colon b \in B\}$. In such a case we say that x is an I-dispersion point of the set B from the right. Similarly the left sided I-dispersion of B at x is defined.

Let $f\colon R \to R$ be a Baire function. If in the definitions of $W_-(f,x)$, $W_+(f,x)$ and $A(f)$ we replace dispersion in the sense of measure by I-dispersion we obtain definitions of $I\text{-}W_+(f,x)$, $I\text{-}W_-(f,x)$ and $I\text{-}A(f)$, respectively.

In [4] it is shown that in the sense of measure the sets $A(f)$ are σ-porous. In the sense of category, the sets $I\text{-}A(f)$ are σ-well porous [6], i.e. they satisfy the following

Definition. A set B is well porous at the point x if

$$\underline{p}(x,B) \overset{\text{def.}}{=} \max\left(\liminf_{\delta \to 0^+} \frac{\gamma_-(x,B,\delta)}{\delta}, \limsup_{\delta \to 0^+} \frac{\gamma_+(x,B,\delta)}{\delta} \right) > 0 ,$$

where $\gamma_+(x,B,\delta)$, resp. $\gamma_-(x,B,\delta)$ denotes the length of the longest open interval contained in $(x,x+\delta) \setminus B$, resp. $(x-\delta,x) \setminus B$. A set B is called well porous if it is well porous at each of its points, and it is called σ-well porous if it is a countable union of well porous sets.

The notion of well porosity is inspired by the following

Lemma. E.Łazarow [1] ,comp. [3] Thm.44 . Let G be an open set. Then $I\text{-}d_+(0,G) = 0$ if and only if for every positive integer n there exist a positive integer k and a positive number $\delta > 0$ such, that for every $h \in (0,\delta)$ and every $i \in \{1,\dots,n\}$ there exists a positive integer $j \in \{1,\dots,k\}$ satisfying the equality

$$\left(\frac{i-1}{n} + \frac{j-1}{nk}h, \frac{i-1}{n} + \frac{j}{nk}h\right) \cap G = \emptyset .$$

The main result of this paper is the following

Theorem. There exists a function $f:R \to R$ such that $A(f)$ is not σ-well porous.

Proof. We construct a set B such that the asymmetry set of its characteristic function is not σ-well porous. By C we denote a Cantor-like set constructed inductively in the following way. In the k-th step we delete from $[0,1]$ a finite number of pairwise disjoint open intervals called D-intervals of order k. The intervals that remain after k steps are called the R-intervals of order k. Any R-interval is closed.

Step 1. Let us choose the interval $\left(\frac{1}{4}, \frac{3}{4}\right)$ as the system of all D-intervals of order 1 and the intervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{3}{4}, 1\right]$ as the system of all R-intervals of order 1.

Inductive step. Let k be a positive integer. Let T be an R-interval of order k and let d_k denote the length of T. As the system of all D-intervals of order $k+1$ in T let us choose $k+1$ open intervals from T each of length $d_k \frac{1}{k+2}$ and such that the complement in T of the union of these intervals has $k+2$ components each of length $d_k \left(\frac{1}{k+2}\right)^2$. These components are the R-intervals of order $k+1$ in T. Let C be the complement in $[0,1]$ of the union of all D-intervals. If $D = (a,b)$ is a D-interval of order k, then let $B_D = (b-d_k, b)$ where d_k denotes the length of the R-interval of order k. Let B be the union of all intervals B_D. It is easy to verify that if $x \in C \setminus \{0\}$ then x is a dispersion point of the set B from the right and x is not a dispersion point of the set B from the left. We have that $A(\chi_B) = (C \setminus \{0\}) \cup E$ where E denotes the set of all left ends of intervals B_D. We show that $A(\chi_B)$ is not σ-well porous. Because E is countable it is sufficient to show that C is not σ-well porous. Assume on the contrary that C is σ-well porous. Then $C = \bigcup_{n=1}^{\infty} E_n$ where E_n is well porous for $n=1,2,\ldots$. As is done in [5] we will define a sequence $\{C_n\}_{n=1}^{\infty}$ of nonempty perfect sets such that $C_{n+1} \subseteq C_n \subseteq C$ and $C_n \cap E_n = \emptyset$ for $n=1,2,\ldots$. The existence of such a sequence yields a contradiction because it implies the existence of a point $x \in \bigcap_{n=1}^{\infty} C_n \subseteq C$ which does not belong to $\bigcup_{n=1}^{\infty} E_n = C$. Define the sets C_n by induction.

1. If $\overline{E}_1 \neq C$, then there exists an R-interval T such that $T \cap E_1 = \emptyset$. Let $C_1 = T \cap C$. If $\overline{E}_1 = C$, then for each

positive integer k and for each D-interval $T = (a,b)$ of order k let $\tilde{T} = \left(a - d_k/3 \,,\, b + d_k/3\right)$, where d_k denotes the length of the R-intervals of order k. Now we define C_1 as the complement in $[0,1]$ of the union of all intervals $\tilde{T}$. It is easy to verify that for all $x \in C_1$

$$\underline{p}\,(x,E_1) = \underline{p}\,(x,\overline{E}_1) = \underline{p}\,(x,C) = 0 \,.$$

Hence $C_1 \cap E_1 = \emptyset$.

2. We observe that the perfect set C_1 has all the properties which are sufficient to construct / in an analogous way / a set $C_2 \subseteq C_1$ and, inductively, a set $C_{n+1} \subseteq C_n$ for a positive integer n.

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