TOPICAL SURVEY

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Quasi-continuity

Introduction

The quasi-continuous mappings introduced by Kempisty for real functions of several real variables have been intensively studied. There are various reasons for the interest in this study. Perhaps the following two are the main ones. The first is a relatively good connection between the continuity and quasi-continuity in spite of the generality of the latter. The second is a deep connection of quasi-continuity with mathematical analysis and topology. There are also some probabilistic connections.

The results concerning quasi-continuity are scattered in the literature. The survey papers [PT 1], [PT 5] of Z. Piotrowski contain among other things also various interesting results of this kind, but they are not directly devoted to this field.

In the present paper we would like to give a survey of results about quasi-continuous mappings. We include also results about quasi-continuous multifunctions.

It is necessary to say that we do not present the list of all results comparing quasi-continuity with the immense number of various generalized continuity notions. Of course, it is not possible to avoid some generalized continuity notions which are closely related to quasi-continuity such as α-continuity [NO 2], somewhat continuity [GH] and cliquishness [TH].

The proofs are included usually in cases when the assertions are more general than those which appear in the literature or when, according to our opinion, the result is not known. Otherwise the reader is referred to the corresponding papers.

We are aware of the fact that various results about quasi-continuity which may be of interest are not included in this paper. Nevertheless, we hope that those which are included will give a comprehensive information concerning quasi-continuity.

1. Various approaches to the notion of quasi-continuity

1.1 Quasi-continuous functions and multifunctions

The words mapping and function have, throughout this paper, the same meaning and both mean a single-valued mapping. If a multi-valued mapping is considered then it is said explicitly. The word multifunction has the same meaning as multi-valued mapping. If nothing else is said, X, Y denote topological spaces. The functions (multifunctions) considered are defined on X and assume their values in Y (in $P(Y) - \emptyset$ where $P(Y)$ is the power set of Y). We denote functions as a rule by f, g, h, etc. while multifunctions are denoted by capital letters F, G, H etc. In case of multifunctions we write simply $F : X \to Y$ instead of $F : X \to P(Y) - \emptyset$. The symbol $f : X \to Y$ (for a function f) has its usual meaning. If $F : X \to Y$ is a multifunction then for $A \subset Y$ we denote

$$F^+(A) = \{x \in X : F(x) \subset A\}, \quad F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}.$$

A function $f : X \to Y$ may be considered as a multifunction assigning to $x \in X$ the singleton $\{f(x)\}$. Usually we identify $\{f(x)\}$ with $f(x)$. In this case we have for $A \subset Y$

$$f^+(A) = f^-(A) = f^{-1}(A) = \{x : f(x) \in A\}.$$

If $A \subset X$ we use the notation $A^0, \bar{A}$ for the interior and the closure of A, respectively. The symbol $Fr(A)$ denotes the frontier of the set A, i.e. the set of those points each neighborhood of which has nonempty intersection with both A and $X - A$.

The definition of quasi-continuity of $f : X \to Y$ was given for the case $X = R_n, Y = R$ by S. Kempisty. Nevertheless, the funtion of two variables being quasi-continuous under the assumption that it is continuous in each variable separately was mentioned by Volterra (see Baire [BA]). Kempisty's definition for general topological spaces may be reformulated [NE 2] in the following way.

1.1.1 A mapping $f : X \to Y$ is quasi-continuous at $p \in X$ if for any U, V open such that $p \in U, f(p) \in V$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$. It is called quasi-continuous if it is quasi-continuous at any $x \in X$.

Evidently any continuous mapping is quasi-continuous. The converse is of course not true. Any monotone left or right continuous function $f : R \to R$ is quasi-continuous. There are very general mappings $f : R \to R$ which are quasi-continuous. They will be discussed in the section 7.

1.1.2 A multivalued mapping $F : X \to Y$ is said to be upper (lower) quasi-continuous at $p \in X$ ([NE 3], [PO 1]) if for any $V \subset Y, V$ open such that $F(p) \subset$

V $(F(p) \cap V \neq \emptyset)$ and for any U open containing p there exists $G \neq \emptyset, G \subset U$, open such that $F(x) \subset V$ $(F(x) \cap V \neq \emptyset)$ for any $x \in G$. It is said to be upper (lower) quasi-continuous if it is upper (lower) quasi-continuous at any $x \in X$.

It is easy to see that in case of a single-valued mapping the notion of upper quasi-continuity and lower quasi-continuity coincide with the quasi-continuity.

Another obvious connection in the multi-valued case is between so-called upper (lower) semi-continuous ([BE]) functions and continuous functions. We will omit the prefix "semi" in this paper so we will use the terms upper continuous and lower continuous. The reason is that "semi" is usually used in another context. So let us recall the definition.

1.1.3 A multi-valued mapping $F : X \to Y$ is said to be upper (lower) continuous at $p \in X$ if for any V open, $V \supset F(p)$ $(V \cap F(p) \neq \emptyset)$ there exists a neighborhood U of p such that $F(x) \subset V$ $(F(x) \cap V \neq \emptyset)$ for any $x \in U$. It is called continuous at p if it is both upper and lower continuous.

Evidently, the upper and lower continuity as well as the continuity at $p \in X$ in case of a single-valued mapping coincide with the usual notion of continuity.

Of course, any upper (lower) continuous multifunction is upper (lower) quasi-continuous while the converse is not true.

1.2 Equivalent definitions of quasi-continuity

In his paper [BL] W.W. Bledsoe introduced for functions of a real variable whose ranges are metric spaces the notion of a neighborly function. If we generalize it for $f : X \to Y$ where both X and Y are metric spaces, we get the following:

1.2.1 A function $f : X \to Y$ where X, Y are metric spaces with the metrics ρ, ρ', respectively, is said to be neighborly at $x \in X$ if for any $\varepsilon > 0$ there exists an open sphere $S \subset X$ such that

$$\rho(x, y) + \rho'(f(x), f(y)) < \varepsilon$$

for any $y \in S$.

S. Marcus [MC 1] has proved that the notions of neighborly and quasi-continuous are equivalent.

In 1961 N. Levine [LE 1] introduced the notion of a semi-continuous function, using the notion of a semi-open set.

1.2.2 A subset A of a topological space X is said to be semi-open if $A \subset \overline{A^0}$. For reasons which will be immediately clear (see 1.2.4) we will use the word quasi-open instead of semi-open.

1.2.3 A function $f : X \to Y$ is called semi-continuous if for any $V \subset Y, V$ open, the set $f^{-1}(V)$ is semi-open (i.e. quasi-open in our terminology).

In this connection one can consider multifunctions for which $F^+(V)$ is quasi-open whenever V is open. Similarly one can consider those multifunctions for which $F^-(V)$ is quasi-open whenever V is open.

In [NA 1] the following was proved.

1.2.4 A single-valued mapping $f : X \to Y$ is quasi-continuous if and only if it is semi-continuous (in the sense of 1.2.3). The above result is a special case of any of the following assertions.

1.2.5 A multifunction is upper (lower) quasi-continuous if $F^+(V)$ $(F^-(V))$ is quasi-open for any open $V \subset Y$.

Proof. We give the proof for the "lower" case. So let F be lower quasi-continuous. Let $V \subset Y$ be open. Let $p \in F^-(V)$ be an arbitrary point and U any neighborhood of p. By the lower quasi-continuity of F at p there is G open, $G \neq \emptyset$ such that $G \subset U$ and $F(x) \cap V \neq \emptyset$ for any $x \in G$. So $G \subset F^-(V)$ and we have $p \in (F^-(V))^0$. Thus $F^-(V)$ is quasi-open. Conversely, let $F^-(V)$ be quasi-open for any $V \subset Y, V$ open. Let $p \in X$ and let V be open such that $F(p) \cap V \neq \emptyset$ and U open such that $p \in U$. Since U is open and $F^-(V)$ quasi-open, $F^{-1}(V) \cap U$ is quasi-open ([LE 1]). Moreover, since $F^-(V) \cap U \neq \emptyset$, we have $(F^-(V) \cap U)^0 \neq \emptyset$. Putting $G = (F^-(V) \cap U)^0$ we have $F(x) \cap V \neq \emptyset$ for any $x \in G$. The lower quasi-continuity at p is proved.

As one may observe we did not define quasi-continuity of a multifunction. A natural definition is the following:

1.2.6 A multifunction $F : X \to Y$ is said to be quasi-continuous at $p \in X$ if for any open sets V, W such that $F(p) \subset V, F(p) \cap W \neq \emptyset$ and for any neighborhood U of p there exists a nonempty open set G such that $G \subset U$ and for any $x \in G$ we have $F(x) \subset V$ and $F(x) \cap W \neq \emptyset$.

It is evident that a quasi-continuous multifunction is both upper and lower quasi-continuous. The converse is not true as the following example shows.

1.2.7 Consider R with the usual topology and define $F : R \to R$ as follows:

$$F(x) = \begin{cases} [0,1] & \text{if} \quad x < 0 \\ [-1, -\frac{1}{2}] \cup [0,1] & \text{if} \quad x = 0 \\ [-1,0] & \text{if} \quad x > 0. \end{cases}$$

Then F is both upper and lower quasi-continuous but not quasi-continuous.

Note that in the case of a single-valued mapping the definition 1.2.6 coincides with the usual quasi-continuity.

1.3 Quasi-continuity and Vietoris topology

Given a topological space Y we may consider on the power set $P(Y)$ the finite or Vietoris topology [MH]. To describe it denote for any positive integer n by $\langle U_1, \ldots, U_n \rangle$, where $U_i (i = 1, \ldots, n)$ are open sets in Y, the collection

$$\{E \in P(Y) : E \subset \bigcup_{i=1}^{n} E_i,\ E_i \cap U_i \neq \emptyset\ \ (i = 1, \ldots, n)\}.$$

1.3.1 The topology on $P(Y)$ generated by the base $\{\langle U_1, \ldots, U_n \rangle, n = 1, 2, \ldots\}$ is called the finite or Vietoris topology on $P(Y)$.

1.3.2 The topology generated by the base (subbase) of all sets

$$\{E \in P(X) : E \subset U\} \qquad (\{E \in P(X) : E \cap U \neq \emptyset\})$$

where U runs through all open sets of Y, is called upper (lower) Vietoris topology or upper (lower) semifinite topology. Given a multifunction $F : X \to Y$ we may consider F as a single-valued mapping defined on X with values in $P(Y) - \emptyset$ and we can consider continuity of F relative to the Vietoris topology or the lower or upper Vietoris topology on $P(Y)$. The following result is well known ([MH]):

1.3.3 A multifunction $F : X \to Y$ is upper (lower) continuous (definition 1.1.3) if and only if it is continuous as a single-valued mapping from the topological space X into $P(Y)$ with the upper (lower) Vietoris topology. It is continuous exactly if the corresponding single-valued mapping is continuous in the Vietoris topology.

Now we consider quasi-continuity of a multi-valued mapping $F : X \to Y$ in such a way that we consider F as a single-valued mapping into $P(Y)$ and the Vietoris topology on $P(Y)$. However, the situation is different from that described in 1.3.3.

If $F : X \to Y$ is quasi-continuous when considered as a single-valued mapping into $P(Y)$ then it is evidently quasi-continuous in the sense of 1.2.6. The converse is not true.

1.3.4 ([EN]) Let $X = A \cup \{0\}$ where $A = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$. Consider the natural topology on X. Let A_i for $i = 1, 2, 3$ be disjoint subsets of A such that $A = A_1 \cup A_2 \cup A_3$ and $0 \in \bar{A}_i$ for $i = 1, 2, 3$. Define the multifunction $F : R \to R$ as follows:

$$F(0) = \{-1, 0, 1\}$$

$$F(x) = \begin{cases} \{1\} & \text{if} \quad x \in A_1 \\ \{0\} & \text{if} \quad x \in A_2 \\ \{-1\} & \text{if} \quad x \in A_3 \end{cases}$$

Now let G_1, G_2 be open such that $F(0) \subset G_1$ and $F(0) \cap G_2 \neq \emptyset$. Let U be any neighborhood of 0. Evidently $F(\frac{1}{n}) \subset F(0) \subset G_1$ for $n = 1, 2, \ldots$. Since $F(0) \cap G_2 \neq \emptyset, G_2$ contains at least one of the elements of the set $\{-1, 0, 1\}$. Without loss of generality we may suppose that $0 \in F(0) \cap G_2$. Choosing n_0 sufficiently large and such that $\frac{1}{n_0} \in A_2$ we obtain a nonempty subset $V = \{\frac{1}{n_0}\} \subset U$ and evidently

$$F(\frac{1}{n_0}) \subset F(G_1), \qquad F(\frac{1}{n_0}) \cap G_2 \neq \emptyset.$$

Thus F is quasi-continuous at 0. The quasi-continuity at any other point $x \in A$ is evident.

Choosing $G_1 = (-\frac{3}{2}, -\frac{1}{2}), G_2 = (-\frac{1}{2}, \frac{1}{2}), G_3 = (\frac{1}{2}, \frac{3}{2})$ we have $F(0) \in \langle G_1, G_2, G_3 \rangle$. But if $E \subset X$ is any set such that $F(x) \epsilon \langle G_1, G_2, G_3 \rangle$ for any $x \in E$, then $E = \{0\}$ which is not an open set. Thus F is not quasi-continuous in the Vietoris topology.

One can easily see that the following holds true.

1.3.5. A multivalued mapping $F : X \to Y$ is upper quasi-continuous if and only if it is upper quasi-continuous as a single-valued mapping $F : X \to P(Y)$.

Note that there is no analogy to 1.3.5 for the lower quasi-continuity. This is easily seen from 1.3.4.

Recall that a space X is said to be extremally disconnected if the closure of every open set $G \subset X$ is open. The following result gives a condition under which various types of quasi-continuity are identical.

1.3.6 ([EN]) Let X be an extremally disconnected space. Then for any multivalued mapping $F : X \to Y$ and any topological space Y the simultaneous upper and lower quasi-continuity, quasi-continuity and quasi-continuity in Vietoris topology coincide.

In the paper mentioned above also the converse of 1.3.6 under general conditions is proved.

1.3.7 Let X be dense in itself. If for any multifunction $F : X \to Y$ and any topological space Y at least two different types of quasi-continuity mentioned in 1.3.6 are identical, then X is extremally disconnected.

It is immediate that the following implications are valid and in general none of them can be reversed.

1.3.8 Let $F : X \to Y$ be a multifunction. Then

a) F is lower (upper) continuous $\Rightarrow$ F is lower (upper) quasi-continuous

b) F is continuous $\Rightarrow$ $\underline{F}$ is quasi-continuous

c) F is continuous $\Rightarrow$ F is quasi-continuous in the Vietoris topology

A necessary and sufficient condition for reversing the implications a), b), c) is included in

1.3.9 A necessary and sufficient condition for reversing the implication a) in 1.3.8 is the following: For any closed set $Z \subset X$ the restriction $F \mid Z$ is lower (upper) quasi-continuous. The quasi-continuity of $F \mid Z$ for any closed set Z guarantees the reversing of b). The situation in the case c) is analogous.

Proof. Since the cases b) and c) follow from a) and from the known relations between quasi-continuity, upper quasi-continuity and lower quasi-continuity as well as the quasi-continuity in Vietoris topology, we investigate only the case a). The upper case may be reduced to single-valued functions (see 1.3.5 and 1.3.3); but the single valued case is known [MT]. So we prove the assertion for the lower case. Suppose F not to be lower continuous. We have to show that there exists a closed set Z such that $F \mid Z$ is not lower quasi-continuous. Let $p \in X$ be a point at which F is not lower continuous. Then there is V open such that $V \cap F(p) \neq \emptyset$ and in any neighborhood U of p there is x such that $F(x) \cap V = \emptyset$. We choose in any neighborhood U of p a point y such that $F(y) \cap V = \emptyset$. Let A be the set of all such points. Put $Z = \bar{A}$. Evidently $p \in Z$. Let U_1 be any open set in Z containing p. Then $U_1 = U \cap Z$ where U is open in X. Now, let $G \subset U_1$ be a nonempty open set in U_1. Then $G \subset Z$. If $G \neq \{p\}$ then G contains a point y in A and we have $F(y) \cap V = \emptyset$. But the case $G = \{p\}$ is impossible because we would have $G = V \cap Z$ where V is open and contains p and thus a point different from p and belonging to A, hence to Z. So $F \mid Z$ is not lower quasi-continuous at p. Since there exists a quasi-continuous mapping which is not continuous, we see from 1.3.8 that quasi-continuity is not a hereditary property with respect to closed sets.

But the following is well known.

1.3.10 Let $F : X \to Y$ be a quasi-continuous (upper quasi-continuous, lower quasi-continuous) multifunction and $G \subset X$ an open set. Then $F \mid G$ is quasi-continuous (upper quasi-continuous, lower quasi-continuous).

1.3.11 The assertion 1.3.10 holds true if the open set G is replaced by a dense set.

2. Continuity types closely related to quasi-continuity

2.1 Somewhat continuity

Somewhat continuous mappings were introduced in [GH].

2.1.1 A mapping $f : X \to Y$ is said to be somewhat continuous if for any open $V \subset Y$ such that $f^{-1}(V) \neq \emptyset$ we have $(f^{-1}(V))^0 \neq \emptyset$.

As can immediately be seen, any quasi-continuous mapping is somewhat continuous. The converse is not true.

2.1.2 Example. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \quad x \text{ is rational} \\ 1 & \text{if } x < 0, \quad x \text{ is irrational} \\ 0 & \text{if } x \in [0,1] \\ 1 & \text{if } x \in (1,\infty) \end{cases}$$

Then f is somewhat continuous but not quasi-continuous. The restriction of a somewhat continuous mapping need not be somewhat continuous even in the case when it is a restriction to an open set. As an example we can take f from 2.1.2 with the open set $(-\infty, 0)$. But the following assertion is useful and gives a connection between somewhat continuity and quasi-continuity.

2.1.3 ([NE 2]) A mapping $f : X \to Y$ is quasi-continuous if and only if there exists a basis $\mathcal{B}$ of open sets such that $f \mid B$ is somewhat continuous for every $B \in \mathcal{B}$.

A useful characterization of somewhat continuity is the following.

2.1.4 ([GH]) A mapping $f : X \to Y$ is somewhat continuous if and only if for any dense set $D \subset X$ the set $f(D)$ is dense in $f(X)$.

As an easy consequence of 2.1.3 and 2.1.4 we have

2.1.5 A mapping $f : X \to Y$ is quasi-continuous if and only if for any dense $D \subset X$ the set $f(D \cap G)$ is dense in $f(G)$ for any G open.

The following is a natural extension of somewhat continuity to multifunctions.

2.1.6 ([NE 3]) A multifunction $F : X \to Y$ is upper (lower) somewhat continuous if for any open $V \subset Y$ for which $F^+(V) \neq \emptyset$ ($F^-(V) \neq \emptyset$) we have $(F^+(V))^0 \neq \emptyset$ ($(F^-(V))^0 \neq \emptyset$).

In case of a single-valued mapping both the upper and the lower somewhat continuity coincide with the somewhat continuity.

Similarly, the upper somewhat continuity of $F : X \to Y$ may be characterized as the somewhat continuity of $F : X \to P(Y)$ when F is considered as a single-valued mapping and the topology on $P(Y)$ is the upper Vietoris topology. The situation is different with the lower somewhat continuity. Example 1.3.4 may serve to show that F is lower somewhat continuous but considered as a single valued mapping it is not somewhat continuous in the lower Vietoris topology on $P(Y)$.

A natural definition of somewhat continuity for a multifunction $F : X \to Y$ may be stated as follows:

2.1.7 ([EN]) A multifunction $F : X \to Y$ is called somewhat continuous if for any two open sets V, W such that $F^+(V) \cap F^-(W) \neq \emptyset$ we have $(F^+(V) \cap F^-(W))^0 \neq \emptyset$.

The above type of somewhat continuity again cannot be characterized by means of the Vietoris topology.

To formulate 2.1.3 and 2.1.4 for multifunctions we need the notions of upper and lower density of a collection of sets.

2.1.8 ([NE 10]) A collection $\mathcal{A}$ of subsets of a topological space Y is said to be upper (lower) dense in a collection $\mathcal{B}$ of subsets of Y if for any $B \in \mathcal{B}$ and for any G open such that $G \supset B$ $(G \cap B \neq \emptyset)$ there exists $A \in \mathcal{A}$ such that $G \supset A$ $(G \cap A \neq \emptyset)$.

It is easy to see that the upper density of $\mathcal{A}$ in $\mathcal{B}$ is the density of $\mathcal{A}$ in $\mathcal{B}$ in the upper Vietoris topology. Since the upper somewhat continuity of a multifunction $F : X \to Y$ is the upper somewhat continuity in Vietoris topology when F is considered as a single-valued mapping into $P(Y)$, we obtain from 2.1.4 and 2.1.3 the following results ([NE 10]).

2.1.9 A multifunction $F : X \to Y$ is upper quasi-continuous if and only if its restriction $F \mid B$ is upper somewhat continuous for any set B belonging to a base $\mathcal{B}$ of open sets in X.

2.1.10 A multifunction $F : X \to Y$ is upper somewhat continuous if and only if for any dense set $D \subset X$ the collection $\{F(x) : x \in D\}$ is upper dense in $\{F(x) : x \in X\}$.

2.1.11 A multifunction $F : X \to Y$ is upper quasi-continuous if and only if for any dense $D \subset X$ the set $\{F(x) : x \in D \cap G\}$ is upper dense in $\{F(x) : x \in G\}$ for any open set $G \subset X$.

The analogues of 2.1.9, 2.1.10, 2.1.11 for the lower case may be obtained directly. Since their proofs are similar to the single-valued case, we omit them. The same type of results may be obtained for somewhat continuity of multifunctions.

2.1.12 Remark. Since the somewhat continuity of a mapping does not coincide with its quasi-continuity, it is evidently different in general from its continuity. Nevertheless if f is a linear somewhat continuous mapping from a linear topological space X into a linear topological space Y then it is continuous [PT 6]. (See also [EW 5] for related results.)

2.2 Strong and weak quasi-continuity

Denote $Q(X)$ the collection of all quasi-open sets in X and $G(X)$ the topology on X. As we have seen, $G(X) \subset Q(X)$ and in general $G(X) \neq Q(X)$. In general $Q(X)$ is not a topology. A simple result shows that the case when $Q(X)$ is a topology is rather exceptional.

2.2.1 ([NS]) $Q(X)$ is a topology if and only if the space X is extremally disconnected.

A class of sets closely related to $Q(X)$ has been studied [NS]. They are usually called α-sets.

2.2.2 A set $A \subset X$ is said to be an α-set if $A \subset (\overline{(A^0)})^0$. Denoting by $\alpha(X)$ the collection of all α-sets in X we have

$$G(X) \subset \alpha(X) \subset Q(X).$$

It is not difficult to show that the above inclusions are in general strict. We will see later that $G(R) \neq \alpha(R) \neq Q(R)$.

While $Q(X)$ is in general not a topology we have the following result.

2.2.3 ([NS]) If X is an arbitrary topological space then $\alpha(X)$ is a topology.

2.2.4 A function $f : X \to Y$ is said to be α-continuous (or strongly quasi-continuous [NO 1]) if it is continuous with respect to the topology $\alpha(X)$ on X.

The following implications are obvious:

2.2.5 Continuity $\Rightarrow$ α-continuity $\Rightarrow$ quasi-continuity. In general none of the implications can be reversed.

2.2.6 ([NO 2]) Let $X = \{a, b, c\}$ and let $G(X) = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $Y = X$ with the topology $G(Y) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Y\}$. Then the identity function is strongly quasi-continuous but not continuous.

2.2.7 Let $X = Y = R$ with the usual topology. Then $f : R \to R$ defined as $f(x) = 0$, if $x \leq 0$, and $f(x) = 1$, if $x > 0$, is quasi-continuous but not α-continuous.

Every real function which is α-continuous is continuous as the following result shows:

2.2.8 Let X, Y be topological spaces, Y regular. Then any $f : X \to Y$ which is α-continuous is also continuous.

This assertion follows from a more general result which will be proved for multifunctions.

2.2.9 A multifunction $F : X \to Y$ is called α-continuous (α-upper continuous, α-lower continuous) if it is continuous (upper continuous, lower continuous) when the topologies $\alpha(X)$ on X and $G(Y)$ on Y are considered.

Obviously, if f is single-valued then α-upper continuity, α-lower continuity and α-continuity coincide.

2.2.10 ([NE 5]). Let $X = \{a, b, c\}$ and $G(X) = \{\emptyset, \{a\}, X\}$. Let $Y = \{x, y\}$ with the discrete topology. Then $\alpha(X) = \{\emptyset, \{a\}, X, \{a, b\}, \{a, c\}\}$. Define $F : X \to Y$ such that $F(a) = \{x, y\}, F(b) = \{x\}, F(c) = \{y\}$. The space Y is normal, F is α-lower continuous but not lower continuous.

2.2.11 Example ([NE 5]). Let $X = [0, 1], Y = R$. Let $E \in \alpha(X)$ such that $E \notin G(X)$. Put $F(x) = \{1\}$ if $x \in E, F(x) = \{0, 1\}$ if $x \notin E$. Then F is α-upper continuous but not upper continuous.

The next result [NE 5] shows some connection between the lower continuity and α-lower continuity of a multi-function. It involves also upper quasi-continuity.

2.2.12 Let X, Y be topological spaces, Y regular. Let $F : X \to Y$ be an α-lower continuous and upper quasi-continuous multifunction. Then F is lower continuous.

As a corollary of 2.2.12 we obtain 2.2.8.

A result analogous to 2.2.12 may be proved for the α-upper continuity ([NE 5]).

2.2.13 Let X, Y be topological spaces, Y normal. Let a closed-valued multifunction $F : X \to Y$ be α-upper continuous and lower quasi-continuous. Then F is upper continuous.

As a corollary of 2.2.12 and 2.2.13 we have

2.2.14 ([NE 5]). Let X, Y be topological spaces, Y normal. Let $F : X \to Y$ be a closed-valued α-continuous multifunction. Then F is continuous.

Note that results analogous to 2.2.12 - 2.2.14 may be obtained involving some other notions of generalized continuity. We omit these types of results since they are not in the main line of our discussion. We refer the interested reader to [HL].

We omit also discussion concerning the so-called weak quasi-continuity. Nevertheless we will meet this notion in some places of our paper so we introduce the definition which is motivated by the notion of weak continuity (see e.g. [LE 2]).

2.2.15 A mapping $f : X \to Y$ is called weakly quasi-continuous at $p \in X$ if for any open V containing $f(p)$ and any open U containing p there exists a nonempty open set $G \subset U$ such that $f(G) \subset \bar{V}$. It is called weakly quasi-continuous if it is weakly quasi-continuous at any $x \in X$.

We refer e.g. to [NO 3] for some relations between weak continuity and several other continuity notions.

3. Continuity points of quasi-continuous functions

3.1 Continuity points of quasi-continuous functions with values in a space with a base of given cardinality

Denote by $C(f), D(f)$ the sets of all continuity (discontinuity) points of f.

A fundamental result concerning continuity points of quasi-continuous functions is due to N. Levine [LE 1].

3.1.1 If $f : X \to Y$ is a quasi-continuous mapping and Y is second countable, then $D(f)$ is of first category.

A generalization of the above result may be obtained in two directions. The first one is that we consider multi-valued mappings; the second one is that we consider spaces more general than the second-countable spaces.

To this end let us recall the notion of k-Baire space ([HC]).

3.1.2 Let k be an uncountable cardinal number. A topological space X is said to be a k-Baire space if the intersection of any collection of cardinality less than k of open dense sets in X is dense in X.

3.1.3 A set E is said to be of first k-category in X if it can be written as a union of fewer than k nowhere dense subsets of X. E is of second k-category if it is not of first k-category.

From the definitions 3.1.2, 3.1.3 easily follows

3.1.4 The complement of a set of first k-category in a k-Baire space X is dense in X.

Denote by $D_\ell(F)$ and $D_u(F)$ the set of all such points in which the multifunction $F : X \to Y$ is not lower (upper) continuous. Further, put $C_\ell(F) = X - D_\ell(F), C_u(F) = X - D_u(F)$.

The following lemma will be useful ([EL]).

3.1.5 If A is a quasi-open set then $Fr(A)$ is nowhere dense.

Now we are able to prove the following results.

3.1.6 Let X, Y be topological spaces. Let Y possess a base of cardinality less than k. Let $F : X \to Y$ be lower quasi-continuous. Then $D_\ell(F)$ is of first k-category.

3.1.7 Let X, Y be topological spaces. Let Y possess a base of cardinality less than k. Let $F : X \to Y$ be upper quasi-continuous compact valued multifunction. Then the set $D_u(F)$ is of first k-category.

Proof. We give the proof of 3.1.7 only; that of 3.1.6 is similar. Let $\mathcal{V} = \{V_t : t \in T\}$ be a base of Y of cardinality less than k. Let $\mathcal{Z}$ be the collection of all finite unions of sets belonging to $\mathcal{V}$. The collection $\mathcal{Z}$ is of cardinality less than k. Let $p \in D_u(F)$. Then there exists an open set $V \supset F(p)$ such that for every open U containing p there exists $q \in U$ such that $F(q) \cap (Y - V) \neq \emptyset$. Thus $p \in Fr(F^+(V))$. Since $F(p)$ is compact, V may be taken from $\mathcal{Z}$. Thus

$$D_u(F) \subset \bigcup_{Z \in \mathcal{Z}} Fr(F^+(Z)).$$

Since $F^+(Z)$ is quasi-open, by upper quasi-continuity of F, the set $Fr(F^+(Z))$ is nowhere dense by 3.1.5. Thus from the above inclusion we have that $D_u(F)$ is of first k-category.

As a corollary of 3.1.6 and 3.1.7 (see also 3.1.4) we have

3.1.8 Let X, Y be topological spaces, X a k-Baire space and let Y possess a base of cardinality less than k. If the assumptions of 3.1.6 and 3.1.7 are satisfied then the sets $C_\ell(F)$ and $C_u(F)$ are dense in X.

As a special case of 3.1.8 we have

3.1.9 Let $f : X \to Y$ be a quasi-continuous function. Let X be a k-Baire space and let Y have a base of cardinality less than k. Then $C(f)$ is dense in X.

Considering X as the classical Baire space (which is k-Baire space of the first uncountable cardinality k) and Y second countable we obtain from 3.1.8 and 3.1.9 well-known results about continuity points of quasi-continuous functions and multifunctions. One of them is 3.1.1. For the others see [EL].

In connection with upper and lower continuity of multifunctions the following question has been discussed [FT], [KN]: If a multifunction is upper (lower) continuous, what is the structure of its lower (upper) continuity points? Analogous questions have been discussed also for quasi-continuity. As to the last we mention [EW 1], [MD 1].

3.2 Continuity points of quasi-continuous functions with values in uniform spaces

If Y is a metric space then (see [BL]) the result stated in 3.1.1 is true without any assumption on the cardinality of the open base of Y, i.e. the following theorem holds.

3.2.1 Let X be a topological space, Y a metric space. Let $f : X \to Y$ be quasi-continuous. Then $D(f)$ is of first category.

The last result motivates further study of continuity points of quasi - continuous functions and multifunctions with values in uniform spaces.

Since such a study is closely related to that of quasi-continuity points of more general functions, called cliquish functions, we recall the last notion.

3.2.2 ([TH]) A mapping $f : X \to Y$ where (Y, ρ) is a metric space is said to be cliquish at $p \in X$ if for any $\varepsilon > 0$ and any neighborhood U of p there exists a nonempty open set G such that for any $u, v \in G$ we have $\rho(f(u), f(v)) < \varepsilon$. It is said to be cliquish if it is cliquish at any $x \in X$.

Evidently, any quasi-continuous function with values in a metric space is cliquish. Simple examples show that the converse is not true.

Denote $A(f)$ the set of cliquishness of a function $f : X \to Y$. Then $A(f)$ is a closed ([LS]) set and the following generalization of 3.2.1 is true.

3.2.3 ([NA 1], cf. also [MC 1]) Let X be a topological space, Y a metric space. Let $f : X \to Y$. Then the set $A(f) - C(f)$ is of first category in X.

All the above results can be looked at from the point of view of multifunctions and so we prove such a result.

3.2.4 ([EW 2]) A multivalued mapping $F : X \to Y$, where Y is a uniform space with uniformity $\mathcal{U}$, is said to be cliquish at $p \in X$ if for every $V \in \mathcal{U}$ and for every neighborhood U of p there exists a nonempty open set $G \subset U$ such that $(F(u) \times F(v)) \cap V \neq \emptyset$ for any $u, v \in G$. It is said to be cliquish if it is cliquish at any $x \in X$.

The above definition coincides with 3.2.2 in case of a single-valued mapping.

For a multi-valued function $F : X \to Y$ denote ([EW 2]) by $C_X(F)$ the set of all points $x \in X$ for which the following is satisfied: For every open set $V \subset Y$ such that $F(x) \subset V$ there exists a neighborhood U of x such that $F(x) \cap V \neq \emptyset$ for any $x \in U$. Further, let $A(F)$ be the set of points where F is cliquish. It can be immediately seen that $A(F)$ is closed.

3.2.5 Let $\mathcal{U}$ be a uniformity on Y which has a base of cardinality less than k. Let $F : X \to Y$ be a compact valued multifunction. Then $A(F) - C_X(F)$ is of first category.

Proof. Let $\{V_t : t \in T\}$ be a base of $\mathcal{U}$ of cardinality less than k. Let H_t be the set of all points $x \in X$ such that any neighborhood U of x contains a point y such that $(F(y) \times F(x)) \cap V_t = \emptyset$. Then we have

$$A(F) - C_X(F) = \bigcup_{t \in T} A(F) \cap H_t.$$

Let G be a nonempty open subset of X. Take $x \in G$ and a set W belonging to the uniformity $\mathcal{U}$ such that $W^2 \subset V_t$. Let $x \in A(F)$. Since F is cliquish at x there exists a nonempty open set $H \subset G$ such that $(F(u) \times F(v)) \cap W \neq \emptyset$ for any $u, v \in H$, and we have $H \cap H_t = \emptyset$. If $x \notin A(F)$ then there is a neighborhood E of x such that $E \cap A(F) = \emptyset$ because $A(F)$ is closed. In any case there is a nonempty open subset G disjoint with $A(F) \cap H_t$. So $A(F) \cap H_t$ is nowhere dense. Since it is true for any $t \in T$, the set $A(F) - C_X(F)$ is of first k-category.

As a corollary we obtain

3.2.6 Let X be a k-Baire space and let Y possess a base of uniformity of cardinality less than k. Let $F : X \to Y$ be lower quasi-continuous multifunction. Then $C_X(F)$ is of first k-category.

It is easy to see that for a single-valued function f we have $C_X(f) = C(f)$ so from 3.2.5 and 3.2.6 we obtain 3.2.1 and 3.2.3 and also the following.

3.2.7 If $f : X \to Y$ is a quasi-continuous function where X is a Baire space, then the set $C(f)$ is dense in X.

There are results about quasi-continuous functions which are in a sense converse to those contained in 3.2.1, 3.2.3 and similar theorems. In this connection we refer the reader to [EI 1], [EI 2], [EI 3]. The following is an example of such a result.

3.2.8 ([EI 1]). Let X, Y be real normed spaces and X a Baire space. Let C, E, A be sets such that $C \subset E \subset A = \bar{A}$ where C is a G_δ and $A - C$ of first category. Then there exists a function $f : X \to Y$ such that $C = C(f), A = A(f)$ and E is the set of quasi-continuity points of f.

3.3 Continuous restrictions of quasi-continuous functions and multifunctions

It is possible to characterize the quasi-continuity of a function f at a point x as follows:

3.3.1 Let X, Y be first countable topological spaces, X a Hausdorff space. Then a function $f : X \to Y$ is quasi-continuous at a point $x \in X$ if and only if there exists a quasi-open set A containing x such that the restriction $f \mid A$ is continuous at x.

The above result follows from more general results which will be given for multifunctions.

3.3.2 Given a collection $\mathcal{K}$ of subsets of Y we say that Y is first countable at $\mathcal{K}$ if for any $K \in \mathcal{K}$ there exists a sequence $\{V_n\}_{n=1}^{\infty}$ of open sets such that $V_n \supset K$ $(n = 1, 2, \ldots)$ and for any open $G \supset K$ there exists n_0 such that $V_{n_0} \subset G$.

3.3.3 ([NN]) Let X be a first countable topological space. Let $F : X \to Y$ be a multifunction and let Y be first countable at the collection $\mathcal{K} = \{F(x) : x \in X\}$. Then F is upper quasi-continuous at a point $x \in X$ if and only if there exists a quasi-open set A containing x such that the restriction $F \mid A$ is upper continuous at x.

As an immediate corollary we obtain

3.3.4 ([NN]) Let X be a first countable Hausdorff space and Y a second countable topological space. Let $F : X \to Y$ be a compact valued multifunction. Then F is upper quasi-continuous at a point $x \in X$ if and only if there exists a quasi-open set A containing x such that $F \mid A$ is upper continuous at x.

Obviously 3.3.1 follows from 3.3.4.

Moreover, it may be shown that an analogous characterization of lower quasi-continuous multifunctions is not possible.

Nevertheless a certain sequential characterization of quasi-continuity is possible even in the case when the characterization by means of the restriction fails. In this connection we mention the following result.

3.3.5 ([NE 6]) Let X, Y be first countable Hausdorff topological spaces. A multifunction $F : X \to Y$ is lower quasi-continuous at $x \in X$ if and only if for any $y \in F(x)$ there is a quasi-open set A containing x such that for any sequence $\{x_n\}_{n=1}^{\infty}, x_n \in A, x_n \to x$ there exists $\{y_n\}_{n=1}^{\infty}, y_n \in F(x_n), y_n \to y$.

In general the quasi-open set A in 3.3.4 depends on the choice of the point $y \in F(x)$. In a special case when the set A is independent of $y \in F(x)$ we may obtain also the characterization of lower quasi-continuity by means of the continuity of restriction. Namely, we have the following

3.3.6 ([NE 6]) Let X, Y be Hausdorff topological spaces. Let $x \in X$. In order that a quasi-open set A containing x exists such that $F \mid A$ is continuous at x, the following condition (C) is necessary and sufficient.

(C) There exists a quasi-open set A containing x (not depending on $y \in F(x)$) such that for any $y \in F(x)$ and any sequence $\{x_n\}_{n=1}^{\infty}, x_n \in A, x_n \to x$, there are $y_n \in F(x_n), y_n \to y$.

4. Quasi-continuity and product spaces

4.1 Separate and joint quasi-continuity

The well-known property of a continuous function $f : X \times Y \to Z$ where X, Y, Z are topological spaces is the continuity of all x-sections where $x \in X$ and all y-sections where $y \in Y$. Here the x-section will be denoted by f_x and means the function $f_x : Y \to Z; f_x(y) = f(x, y)$. In a similar way the y-section $f^y : X \to Z$ is defined. It is also a well-known fact that the continuity of all x-sections where $x \in X$ and all y-sections where $y \in Y$ does not imply the continuity of f. As to the connection between separate and joint continuity we refer the reader to [PT 1]. Further, the continuity (quasi-continuity, somewhat continuity) of all x and y-sections of f is called separate continuity (quasi-continuity, somewhat continuity).

Here the connection between separate and joint quasi-continuity will be discussed. First of all it is necessary to say that the situation is different from that of continuity. The following may serve as an example.

4.1.1 ([MT]). Let $f : (0,1) \times (0,1) \to R$ be defined as

$$f(x,y) = \begin{cases} 1 & \text{if} \quad 0 < x < \frac{1}{2},\ 0 < y < 1 \\ 0 & \text{if} \quad \frac{1}{2} < x < 1,\ 0 < y < 1 \\ 1 & \text{if} \quad x = \frac{1}{2},\ y \in (0,1),\ y \ \textit{rational} \\ 0 & \text{if} \quad x = \frac{1}{2},\ y \in (0,1),\ y \ \textit{irrational} \end{cases}$$

The function f is quasi-continuous but the x-section $f_{\frac{1}{2}}$ is not quasi-continuous. It is not even somewhat continuous.

On the other hand under general conditions on the spaces X, Y, Z the quasi-continuity of x-sections and y-sections implies the quasi-continuity of f. The first result in this direction, where the quasi-continuity of the sections is explicitly used, is due to S. Kempisty ([KP]). The Kempisty's result, roughly speaking, says that a real function $f : (0,1) \times (0,1) \to R$ which has quasi-continuous x-sections and quasi-continuous y-sections is quasi-continuous as a function of two variables. (Note that under the assumption that f_x and f^y are continuous the quasi-continuity of f as a function of two variables was observed earlier.)

There are many generalizations of Kempisty's result ([NE 2], [NE 3], [NP], [BN]).

The following is a general version covering various of those which appear in the literature.

4.1.2 Let X be a k-Baire space, Y a space which possesses at each point a neighborhood which has a base of cardinality less than k and Z a regular space. If $f : X \times Y \to Z$ is separately quasi-continuous, then it is quasi-continuous.

To prove 4.1.2 we give some results related to separate somewhat continuity. Note that the sections F_x, F^y of a multifunction $F : X \times Y \to Z$ are defined in an obvious way. Further, note that the separate somewhat continuity does not imply somewhat continuity even in the case of a function $f : R \times R \to R$ ([NE 7]). But the following is true.

4.1.3 Let X be a k-Baire space, let Y possess a base of cardinality less than k and let Z be a regular space. Let $F : X \times Y \to Z$ be a multifunction such that F_x is lower somewhat continuous for every $x \in X$ and F^y both lower somewhat continuous and upper quasi-continuous for every $y \in Y$. Then Y is lower somewhat continuous.

Proof. Suppose F not to be somewhat continuous. Then there exists an open set $H \subset Z$ such that

$$F^-(H) \neq \emptyset \text{ and } (F^-(H))^0 = \emptyset.$$

So there is a dense set $D \subset X \times Y$ such that for any $(u, v) \in D$ we have $F(u, v) \cap H = \emptyset$. Let $(p, q) \in F^-(H)$ and $z \in F(p, q) \cap H$. Choose H_1 such that

$$z \in H_1 \subset \bar{H}_1 \subset H.$$

Since $z \in F^q(p) \cap H_1$, we have $(F^q)^-(H_1) \neq \emptyset$. So the lower somewhat continuity of F^q implies that a nonempty open set G exists such that for any $x \in G$

$$F^q(x) \cap H_1 \neq \emptyset.$$

Let $\{V_t : t \in T\}$ be a base of Y of cardinality less than k. Put

$$A_t = \{x \in G : F(x, y) \cap H_1 \neq \emptyset \text{ for every } y \in V_t\}.$$

Using ℓ-somewhat continuity of the sections F_x we obtain that

$$G = \bigcup_{t \in T} A_t.$$

To obtain a contradiction with the fact that X is a k-Baire space, it is sufficient to prove that each A_t is nowhere dense. So, let $W \subset G$ be a nonempty open set. Let t be fixed and $(u, v) \in W \times V_t$ be a point such that $F(u, v) \subset Y - H$. The set $Y - \bar{H}_1$ is an open set containing $F(u, v)$. Using the upper quasi-continuity of F^v at u we get a nonempty open set $\tilde{W} \subset W$ such that $F(x, v) \subset Y - \bar{H}_1$ for any $x \in \tilde{W}$. Thus

$$F(x, v) \cap H_1 = F^v(x) \cap H_1 = \emptyset.$$

Since $v \in V_t$, we have $x \notin A_t$. Hence $\tilde{W} \cap A_t = \emptyset$. Thus A_t is nowhere dense. The proof is finished.

In a similar way the following may be proved.

4.1.4 Let X be a k-Baire space and let Y possess a base of cardinality less than k. Let Z be a normal space. Let $F : X \times Y \to Z$ be a closed-valued multifunction such that F_x is upper somewhat continuous for every $x \in X$ and F^y both upper somewhat continuous and lower quasi-continuous for every $y \in Y$. Then F is upper somewhat continuous.

From 4.1.3 and 4.1.4 we obtain results concerning product quasi-continuity.

4.1.5 Let X be a k-Baire space, let Y be a space such that for any $y \in Y$ there exists a neighborhood $V(y)$ possessing a base of cardinality less than k, and let Z be a regular space. Let the multifunction $F : X \times Y \to Z$ be such that F_x is lower quasi-continuous and F^y both upper quasi-continuous and lower quasi-continuous $(x \in X, y \in Y)$. Then F is lower quasi-continuous.

4.1.6 Let X be a k-Baire space, let Y be a space such that for any $y \in Y$ there exists a neighborhood $V(y)$ possessing a base of cardinality less than k, and let Z be a normal space. Let $F : X \times Y \to Z$ be a closed valued multifunction such that F_x is upper quasi-continuous and F^y both upper quasi-continuous and lower quasi-continuous $(x \in X, y \in Y)$. Then F is upper quasi-continuous.

Since the proofs of 4.1.5 and 4.1.6 are similar we give only the proof of 4.1.5.

Proof of 4.1.5. The collection $\{U \times V\}$ where U, V are open in X, Y respectively, is a base in $X \times Y$. From the assumptions it follows that we can take as a base such collection $\{U \times V\}$ where V as a subspace of Y has a base of cardinality less than k. The restriction $F \mid U \times V$ of F to $U \times V$ satisfies on $U \times V$ the assumptions of 4.1.3. So $F \mid U \times V$ is lower somewhat continuous. Using the analogue of 2.1.9 for lower quasi-continuity we see that F is lower quasi-continuous.

4.1.6 If we consider the quasi-continuity of a multifunction (in the sense of Definition 1.2.6) then proofs similar to those of theorems 4.1.3 and 4.1.5 give the following results (for their special case see [EN]).

4.1.7 Let X be a k-Baire space and let Y possess a base of cardinality less than k. Let Z be a normal space. Let $F : X \times Y \to Z$ be a closed-valued multifunction such that

(i) F_x is somewhat continuous for every $x \in X$

(ii) F^y is both upper and lower quasi-continuous for every $y \in Y$.

Then F is somewhat continuous.

4.1.8 Let X be a k-Baire space. Suppose that for every $y \in Y$ there exists a neighborhood possessing a base of cardinality less than k. Let Z be a normal space and $F : X \times Y \to Z$ a closed valued multifunction such that F_x is quasi-continuous for every $x \in X$ and F^y quasi-continuous for every $y \in Y$. Then F is quasi-continuous.

It is easy to see that the condition (i) in 4.1.7 may be weakened in such a way that we assume the somewhat continuity with the exception of a set of first k-category. Some assumptions of the other theorems of this section may be weakened in a similar way.

The case of a real function of n variables which covers the Kempisty theorem is also an easy consequence of our results. We give here a sufficiently general formulation which is useful for applications. Of course, it is not the most general version obtainable from the above results.

4.1.9 If X_i $(i = 1, 2, \ldots, n)$ are second countable Baire spaces and f a function on $X_1 \times X_2 \times \ldots \times X_n$ to a metric space Y such that for any $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ the sections f_{x_i} $(i = 1, 2, \ldots, n)$ are quasi-continuous, then f is quasi-continuous.

The question whether the condition of regularity of the range Z in theorems on product quasi-continuity is essential is discussed in [BZ 1]. It is shown that the condition of regularity cannot be replaced by quasi-regularity.

Recall that Z is quasi-regular if for any nonempty open $V \subset Z$ there exists a nonempty open G such that $\bar{G} \subset V$.

Results closely connected with product quasi-continuity are contained in [FD], [GR 1] and [SA 3].

4.2 Symmetrical quasi-continuity

Studying quasi-continuous functions on product spaces Kempisty ([KE]) introduced also a useful notion which he called symmetrical quasi-continuity. We state a generalized version of his definition.

4.2.1 A function $f : X \times Y \to Z$ is called symmetrically quasi-continuous at (p, q) with respect to y if for any neighborhood $U \times V$ of (p, q) where U, V are neighborhoods of p, q, respectively, and for any neighborhood W of $f(p, q)$ there exist a neighborhood H of q such that $H \subset V$ and a nonempty open set $G \subset U$ such that for any $(x, y) \in G \times H$ we have $f(x, y) \in W$. If f is symmetrically quasi-continuous with respect to y at any $(p, q) \in X \times Y$ then it is said to be symmetrically quasi-continuous with respect to y.

The definition of symmetrical quasi-continuity with respect to x is analogous.

If f is symmetrically quasi-continuous both with respect to x and y at $(p, q) \in$

$X \times Y$ it is said to be symmetrically quasi-continuous at (p,q).

If f is symmetrically quasi-continuous at any $(p,q) \in X \times Y$, then it is said to be symmetrically quasi-continuous. One may show easily that if f is symmetrically quasi-continuous then all the x-sections and y-sections are quasi-continuous. The converse is not true even in the case when f_x and f^y are continuous as the following example shows.

4.2.2 ([MT]) Let f be defined by

$$f(x,y) = \begin{cases} \sin \frac{1}{x^2+y^2} & \text{if} \quad x^2+y^2 \neq 0 \\ 0 & \text{if} \quad x^2+y^2 = 0 \end{cases}$$

For the sake of shortness we omit the definition of symmetrical quasi-continuity for multifunctions. Obviously there will be lower and upper cases. In this section we investigate only single-valued functions giving at the end some remarks for multifunctions.

Fundamental theorems concerning symmetrical quasi-continuity on various levels of generality have been stated successively in [KE], [MT], [BN], [PT 2], [PT 3], [LT]. We prove the following version to cover the mentioned results.

4.2.3 Let X be a k-Baire space, let Y possess at any $y \in Y$ a basis of neighborhoods of cardinality less than k and let Z be a regular space. Let $f : X \times Y \to Z$ be such that f^y is quasi-continuous for any y belonging to a dense set $H \subset Y$. Let f_x be continuous on H for any $x \in X$. Then f is symmetrically quasi-continuous with respect to y at any $(p,q) \in X \times Y$ for which $q \in H$.

To prove 4.2.3 we use the notion of symmetrical somewhat continuity.

4.2.4 A mapping $f : X \times Y \to Z$ is said to be symmetrically somewhat continuous with respect to y at a point (p,q) if for any neighborhood V of $f(p,q)$ there exists a nonempty open set $G \subset X$ and a neighborhood H of q such that $G \times H \subset f^{-1}(V)$. It is said to be symmetrically somewhat continuous with respect to y if it is symmetrically somewhat continuous with respect to y at any $(p,q) \in X \times Y$.

The symmetrical somewhat continuity with respect to x is defined in an analogous way. If f is symmetrically somewhat continuous (at (p,q)) with respect to both x and y, it is said to be symmetrically somewhat continuous (at (p,q)).

The proofs of the following assertions are simple and therefore they are omitted.

4.2.5 If f is symmetrically quasi-continuous with respect to x (with respect to y, symmetrically quasi-continuous) then it is symmetrically somewhat continuous with respect to x (with respect to y, symmetrically somewhat continuous).

4.2.6 If f is symmetrically quasi-continuous with respect to x (with respect to y, symmetrically quasi-continuous) and $G \subset X \times Y$ is an open set then $f \mid G$ is symmetrically quasi-continuous with respect to x (with respect to y, symmetrically quasi-continuous).

4.2.7 A mapping $f : X \times Y \to Z$ is symmetrically quasi-continuous with respect to x (with respect to y, symmetrically quasi-continuous) if and only if there exists a basis B of open sets in $X \times Y$ such that $f \mid B$ is symmetrically somewhat continuous with respect to x (with respect to y, symmetrically somewhat continuous) for each $B \in \mathcal{B}$.

It is a matter of easy examples to show that the somewhat continuity does not imply the symmetrical somewhat continuity as well as the symmetrical somewhat continuity does not imply quasi-continuity.

Proof of Theorem 4.2.3. It is sufficient to prove that f is symmetrically somewhat continuous with respect to y at any (p,q) where $q \in H$ and then to apply 4.2.7. So suppose that f is not symmetrically somewhat continuous with respect to y at (p,q) where $q \in H$. Then there is an open set $W \subset Z$ such that $z = f(p,q) \in W$ and that for any $G \times V$ where $G \neq \emptyset$ is open in X and V is a neighborhood of q there exists (u,v) such that $f(u,v) \notin W$. From the regularity of Z we get an open set W_1 such that $z \in W_1 \subset \bar{W}_1 \subset W$. Using quasi-continuity of f^q we have $((f^q)^{-1}(W)) = U \neq \emptyset$. Let $\{V_t : t \in T\}$ be a basis at q of cardinality less than k. Put

$$A_t = \{x \in U : f_x^{-1}(V_t) \subset W_1\}.$$

From the continuity of f_x we obtain immediately $U = \bigcup_{t \in T} A_t$. Now, if we prove that A_t is nowhere dense for any $t \in T$, we obtain that U is of first k-category. It will be a contradiction. So let $G \subset U$ be an open set. According to the assumption there is a point $(u,v) \in G \times V_t$ such that $f(u,v) \notin W$. Using the quasi-continuity of f^v we obtain a nonempty set $E \subset G$ such that $E \cap A_t = \emptyset$. So A_t is nowhere dense.

As a corollary we obtain the following theorem on product quasi-continuity (see [BN] for a more special case).

4.2.8 Let X be a k-Baire space, let Y possess at each point y a base of cardinality less than k and let Z be regular. If f_x is continuous for any $x \in X$ and f^y quasi-continuous for any y belonging to a dense set $H \subset Y$, then f is quasi-continuous.

Proof. Let $(p,q) \in X \times Y$ be any point. Let W be a neighborhood of $f(p,q)$ and $U \times V$ any neighborhood of (p,q). By continuity of f_p we have $q_1 \in V$ such that $f(p,q_1) \in W$. Now it is sufficient to use the symmetrical quasi-continuity with respect to y at (p,q_1) and the proof is finished.

A result for symmetrical quasi-continuity with respect to x, analogous to 4.2.8, is obvious. Hence we have the following well known result (the more general formulation for k-Baire spaces is left to the reader).

4.2.9 Let X, Y be first countable Baire spaces and Z a regular space. Let f be separately continuous. Then f is symmetrically quasi-continuous.

It is also known that the converse to 4.2.9 is not true. As an example we can take 4.2.2.

Now we formulate a theorem on symmetrical quasi-continuity of multifunctions.

4.2.10 Let X be a k-Baire space, let Y have at each point a basis of cardinality less than k and let Z be regular. Let $F : X \times Y \to Z$ be a multifunction such that F_x is continuous for any $x \in X$ and F^y is lower quasi-continuous and upper quasi-continuous for any y belonging to a dense set $H \subset Y$. Then F is lower symmetrically quasi-continuous with respect to y at any (p, q) for which $q \in H$.

Analogous theorems may be formulated and proved for upper symmetrical quasi-continuity when compact valued multifunctions are considered. In this connection we refer the reader to [BZ 2] where slightly weaker variants are proved.

The symmetrical quasi-continuity of a function f (or a multifunction F) has an influence on the set of its continuity points. Results concerning this fact are, in a sense, a continuation of the classical result of Baire [BA] who proved that if $f : [0,1] \times [0,1] \to R$ is separately continuous then there is a residual set $A \subset [0,1]$ and a residual set $B \subset [0,1]$ such that f is continuous on $\{x\} \times [0,1]$ for any $x \in A$ and is also continuous on $[0,1] \times \{y\}$ for every $y \in B$.

For the symmetrically quasi-continuous functions we have weaker but parallel results which appear again on various levels of generality in [KE], [LT], [PT 3], [PT 4], [BB].

As an example we include without proof such a result of Piotrowski [PT 4].

4.2.11 Let X be a topological space, Y a quasi-regular strongly countable complete space and Z a metric space. Let $f : X \times Y \to Z$ be symmetrically quasi-continuous with respect to x. Then the set of continuity points of f which lie on $\{x\} \times Y$ is a dense G_δ subset of $\{x\} \times Y$.

The notion of strongly countably complete space is due to [FL 1]. A space Y is said to be strongly countably complete if there exists a sequence $\{\mathcal{A}_i : i = 1, 2, \dots\}$ of open coverings of Y such that a decreasing sequence $\{F_i\}$ of nonempty closed subsets of Y has a nonempty intersection, provided that each F_i is a subset of a member of $\mathcal{A}_i$.

Since every quasi-regular, strongly countably complete space is a Baire space [PT 4] we obtain from 4.2.9 and 4.2.11 the following corollary.

4.2.12 ([PT 4]) Let X be first countable, Y strongly countably complete quasi-regular and Z a metric space. If $f : X \times Y \to Z$ is such that all its x-sections are quasi-continuous and all its y-sections are continuous then for all $x \in X$ the set of continuity points of f which lie on $\{x\} \times Y$ is a dense G_δ subset in $\{x\} \times Y$.

For a result closely related to 4.2.12 involving cliquish functions we refer to Fudali ([FD]).

5. Quasi-continuity and convergence

5.1 Pointwise, uniform, and quasi-uniform convergence

A standard proof shows that a uniform limit of a sequence of quasi-continuous functions is quasi-continuous.

A simple example of a sequence $\{f_n\}_{n=1}^{\infty}$ where $f_n : [0,1] \to R$, $n = 1, 2, \ldots$, $f_n(x) = x^n$, shows that a sequence of quasi-continuous functions may not converge to a quasi-continuous function. However, the following is a well known result.

5.1.1 ([BL]) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of quasi-continuous functions defined on a topological space X with values in a metric space Y. Let $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in X$. Then the set $D(f)$ of discontinuity points of f is of first category.

As an easy corollary one obtains the following

5.1.2 ([EW 2]) Let $f_n : X \to Y, n = 1, 2, \ldots$ where X is a Baire space and Y a metric space. If for any $x \in X$ $\lim_{n\to\infty} f_n(x) = f(x)$, then f is cliquish.

A condition under which a limit of quasi-continuous functions is quasi-continuous is given in [BR].

Some question about pointwise convergent quasi-continuous multifunctions are answered in [EW 2], [BU], [BC].

The quasi-uniform convergence of quasi-continuous functions was discussed in [DO 1]. Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ of functions defined on X with values in a metric space (Y, ρ) is said to be quasi-uniformly convergent to $f : X \to Y$ provided that it is pointwise convergent to f and moreover for any $x \in X$, any $\varepsilon > 0$, and any positive integer n there exists p such that

$$\min\{\rho(f_n(x), f(x)), \rho(f_{n+1}(x), f(x)), \ldots, \rho(f_{n+p}(x), f(x))\} < \varepsilon.$$

It was proved in [DO 1] that the quasi-uniform convergence does not preserve the quasi-continuity. For the convenience of the reader we introduce the example showing the last mentioned fact.

5.1.3 ([Do 1]) Let for $n = 1, 2, \ldots, f_n(x) = \chi_{[0,\frac{1}{n}]}((-1)^n x)$, where χ_E denotes the characteristic function of the set E. Then $\{f_n\}_{n=1}^{\infty}$ quasi-uniformly converges to $\chi_{\{0\}}$, but $\chi_{\{0\}}$ is not quasi-continuous.

5.2 Quasi-continuity and transfinite convergence

There are interesting connections between quasi-continuity and transfinite convergence. A systematic study of transfinite convergence of quasi-continuous functions and also of the last mentioned connections is contained in [NA 2-6]. A few of these results are mentioned in [PT 5]. Here we give some general results of [NA 2-6] with the emphasis on characterization of locally separable metric spaces by means of transfinite convergence of quasi-continuous functions [NA 4,5].

5.2.1 ([SI]), [NA 6]) Let ω_1 be the first uncountable ordinal number. A transfinite sequence $\{a_\xi\}_{\xi<\omega_1}$ of elements of a topological space Y is said to be convergent to $a \in Y$ if to any neighborhood V of a there exists ξ_0 such that for $\xi > \xi_0$ we have $a_\xi \in V$. A transfinite sequence $\{f_\xi\}_{\xi<\omega_1}$ of functions defined on X and taking values in Y is said to be (pointwise) convergent to $f : X \to Y$ if $\{f_\xi(x)\}_{\xi<\omega_1}$ is convergent to $f(x)$ for any $x \in X$.

The following is a useful lemma (for real functions see [SI], for more general cases [KT], [SA 1], [NA 6]).

5.2.2 Let $\{f_\xi\}_{\xi<\omega_1}$ be a transfinite sequence of functions defined on X and assuming values in a first countable T_1-space Y converging to $f : X \to Y$. Let $S \subset X$ be a countable set. Then there exists $\xi_0 < \omega_1$ such that $f_\xi(x) = f(x)$ for any $\xi \geq \xi_0$ and for any $x \in S$.

A general theorem concerning the preservation of quasi-continuity under transfinite convergence (see [NA 5] for a slightly different formulation) claims:

5.2.3 Let X be locally separable, first countable topological space. Let Y be first countable T_1-space. Let $\{f_\xi\}_{\xi<\omega_1}$ be a transfinite sequence of quasi-continuous functions $f_\xi : X \to Y$ converging to $f : X \to Y$. Then f is quasi-continuous.

Proof. Suppose f not to be quasi-continuous at x_0. Then a neighborhood V of $f(x_0)$ and a neighborhood U of x_0 exist such that for any nonempty open $G \subset U$ there is $x \in G$ with $f(x) \notin V$. The neighborhood U may be supposed to be separable. Let

$$M = \{x : x \in U,\ f(x) \notin V\}.$$

Let D be a countable dense set in U. If $s \in D$, let $\{B_n^s\}, B_n^s \subset U$ $(n = 1, 2, \ldots)$ be a countable basis of neighborhoods of s. There exists $x_n^s \in B_n^s$ such that $x_n^s \in M$.

The set

$$T = \{x_n^s : s \in D,\ n = 1, 2, \ldots\}$$

is a countable dense set in U. Using 5.2.2 we obtain $\xi_0 < \omega_1$ such that

$$f_\xi(x) = f(x) \text{ for any } x \in T \cup \{x_0\} \text{ and any } \xi \geq \xi_0.$$

Using the quasi-continuity of f_{ξ_0} at x_0 we have a nonempty open set $G \subset U$ such that $f_{\xi_0}(x) \in V$ for any $x \in G$. But $G \cap T \neq \emptyset$ so we have $x \in G \cap T \subset M$ such that $f_\xi(x) = f_{\xi_0}(x) \in V$. This is a contradiction.

The local countability in 5.2.3 may be omitted if X is supposed to be strongly locally separable, i.e. a space such that every point possesses a separable neighborhood ([NA 6]).

In case of locally separable metric spaces we have the following characterization.

5.2.4 ([NA 5]) A metric space (X, ρ) is locally separable if and only if for any subspace $Y \subset X$ the following holds: If $\{f_\xi\}_{\xi<\omega_1}$ is a transfinite sequence of real quasi-continuous functions on Y that converges to $f : Y \to R$, then f is quasi-continuous on Y.

Proof. The sufficiency follows from 5.2.3. We give a sketch of proof of necessity (see [NA 5]). Let X be not locally separable. Then there exists ([NT]) an isolated set $M \subset X$ such that M has a condensation point y, i.e. $U \cap M$ is uncountable for any neighborhood U of y. Denote $I_n = \{x \in M : \rho(x, y) < \frac{1}{n}\}$. Now we construct a transfinite sequence $\{x_\xi\}_{\xi<\omega_1}$ in the following way. Choose $x_1 \in M$ arbitrarily. Suppose that $\{x_\eta\}$ is constructed for any $\eta < \xi, \xi < \omega_1, \xi > 1$. If ξ is not a limit number then $\xi = \xi_0 + n$ where ξ_0 is a limit number and n an integer. Then we choose $x_\xi \in I_n, x_\xi \neq x_\eta$ for $\eta < \xi$. If ξ is a limit number we choose any point belonging to M such that $x_\xi \neq x_\eta$ for $\eta < \xi$. Put $Z = \{x_\xi : \xi < \omega_1\} \cup \{y\}$. Then Z is not a locally separable subspace of X. Define for $\xi < \omega_1$

$$f(x) = \begin{cases} 0 & \text{if} \quad x = x_\eta, \quad \eta < \xi \\ 1 & \text{if} \quad x = x_\eta, \quad \eta \geq \xi \\ 1 & \text{if} \quad x = y. \end{cases}$$

We omit the details included in [NA 5] showing that f_ξ are quasi-continous for $\xi < \omega_1$ and the limit function f where $f(x) = \begin{cases} 0 & \text{if} \ x \in Z \\ 1 & \text{if} \ x = y \end{cases}$ is not quasi-continuous.

6. Quasi-continuity and Blumberg sets

6.1 Full Blumberg sets

The close relation between quasi-continuity and Blumberg sets was discovered by J.C. Neugebauer in his paper [NG].

In the paper [BM] of H. Blumberg it was proved that for any real function $f : R \to R$ there exists a dense set $D \subset R$ such that $f \mid D$ is continuous. In general we adopt the following definition.

6.1.1 Let $f : X \to Y$, where X, Y are topological spaces, be a function. A dense set $D \subset X$ is called a Blumberg set for f if $f \mid D$ is continuous. We do not discuss problems concerning the existence of Blumberg sets. For the readers interested in this direction we mention [BG], [WH], [GO], [LV], [HC], [BW], [PS], [AL]. Here only the results giving mutual connections between quasi-continuity and Blumberg sets are dealt with.

To formulate in a general way the mentioned result of J.C. Neugebauer we mention the notion of full Blumberg set.

6.1.2 A set $D \subset X$ is called full Blumberg set for the function f if D is a Blumberg set for f and for every open $G \subset X$ the set $f(G \cap D)$ is dense in $f(G)$.

The result of J.C. Neugebauer states:

6.1.3 A function $f : [0,1] \to R$ possesses a full Blumberg set if and only if f is quasi-continuous.

Note that in [NG] instead of full Blumberg set the so-called strong Blumberg set is considered. But the difference is not essential and the notion of full Blumberg set seems to be more suitable for the considerations in general topological spaces.

The result 6.1.3 was extended for more general spaces by Z. Piotrowski.

6.1.4 ([PT 8]) A function $f : X \to Y$ where Y is a regular topological space possesses a full Blumberg set if and only if it is quasi-continuous.

We will discuss the multi-valued version of 6.1.4.

6.1.5 ([NE 4]) A dense set $D \subset X$ is called an upper (lower) Blumberg set for a multi-valued mapping $F : X \to Y$ if $F \mid D$ is upper (lower) continuous.

6.1.6 ([NE 4]) A set $D \subset X$ is called full upper (lower) Blumberg set for a multifunction $F : X \to Y$ if D is upper (lower) Blumberg set for F and $F(D \cap G)$ is upper (lower) dense in $F(G)$ for any open set $G \subset X$.

From 1.3.10 we obtain

6.1.7 If $F : X \to Y$ is upper (lower) quasi-continuous and $D \subset X$ is an upper (lower) Blumberg set for F then D is full upper (lower) Blumberg set for F.

The converse to 6.1.7 is true neither for upper nor for lower case as the following examples show.

6.1.8 ([NE 4]) Let $X = Y = (-\infty, \infty)$ with the usual topology. Define $F : X \to Y$ as follows:

$$F(x) = \begin{cases} \{0\} \text{ if } x \notin Q, & Q = \{r_n : n = 1, 2, \dots\} \text{ is the set of all rationals} \\ \{0, 1, 2, \dots, n\} \text{ if } x = r_n, n = 1, 2, \dots \end{cases}$$

Then F is not upper quasi-continuous at any $x \in Q$. But $X - Q$ is full upper Blumberg set for F.

If X, Y, Q have the same meaning as above and

$$G(x) = \begin{cases} \{n\} & \text{if } x \in Q,\ x = r_n \\ \{1, 2, \dots\} & \text{if } x \notin Q, \end{cases}$$

then G is not lower quasi-continuous at any $x \in X$ but $X - Q$ if full lower Blumberg set for G.

6.1.9 ([NE 4]) Let $F : X \to Y$ be a multifunction where Y is a regular space. If there exists a full lower Blumberg set D for F such that $F(D \cap G)$ is upper dense in $F(G)$ for any open $G \subset X$, then F is lower quasi-continuous.

6.1.10 Let $F : X \to Y$ be a closed-valued multifunction, where Y is a normal space. If there exists a full upper Blumberg set D for F such that $F(G \cap D)$ is lower dense in $F(G)$ for every G open, then F is upper quasi-continuous.

The proofs of 6.1.9 and 6.1.10 are similar. Since the results imply 6.1.3 and 6.1.4 and since they have other interesting corollaries we give the idea of the proof of 6.1.9.

Proof of 6.1.9. Let $x_0 \in X, U$ open containing $x_0, y \in F(x_0)$ arbitrary. For any open neighborhood V of y let V_1 be such open neighborhood of y that $\overline{V}_1 \subset V$. Since D is a full lower Blumberg set for F there exists $x_1 \in D \cap U$ such that $F(x_1) \cap V_1 \neq \emptyset$. The lower continuity of $F \mid D$ at x_1 implies that there exists a nonempty set $G \subset U$ such that $F(x) \cap V_1 \neq \emptyset$ for any $x \in G \cap D$. Now it is not difficult to prove that $F(x) \cap V \neq \emptyset$ for any $x \in G$ and the lower quasi-continuity of F at x_0 is proved.

Among various corollaries of 6.1.9 and 6.1.10 we have the following two. For the others we refer to [NE 4].

6.1.11 Let $F : X \to Y$ be a multifunction and Y a regular space. Let F have full lower Blumberg set which is simultaneously full upper Blumberg set for F. Then F is lower quasi-continuous.

6.1.12 Let $F : X \to Y$ be a multifunction and Y a regular space. Let F have full lower Blumberg set and let F be upper quasi-continuous. Then F is lower quasi-continuous.

6.2 Simultaneous quasi-continuity and Blumberg sets

The mentioned paper [NG] starts also the study of simultaneous Blumberg sets.

6.2.1 If $f : [0,1] \to [0,1]$ is a bijection then a dense set $D \subset [0,1]$ is called a simultaneous Blumberg set for f if $f(D)$ is dense in $[0,1]$ and if it is a Blumberg set for the inverse function f^{-1}.

The following is a fundamental result relating quasi-continuity to simultaneous Blumberg sets.

6.2.2 ([NG]) Let $f : [0,1] \to [0,1]$ be a quasi-continuous bijection. Then f admits a simultaneous Blumberg set if and only if f^{-1} is quasi-continuous.

An example of quasi-continuous bijection $f : [0,1] \to [0,1]$ such that f^{-1} is not quasi-continuous constructed in [NG] shows that for a quasi-continuous bijection $f : [0,1] \to R$ the simultaneous Blumberg set need not exist. Another example of this type was given by G. Goffman [GO].

A generalization of the notion of simultaneous Blumberg set for a bijection is the notion of simultaneous Blumberg set for a collection of bijections.

6.2.3 ([PT 9]) Let X, Y be topological spaces and let $\mathcal{F} = \{f_t : f_t : X \to Y \ (t \in T)\}$, where T is an index set, be a collection of bijections. A set $D \subset X$ is called a simultaneous Blumberg set for $\mathcal{F}$ if it is a simultaneous Blumberg set for f_t for each $t \in T$.

The following generalization of 6.2.2 is proved in [PT 9].

6.2.4 Let X, Y be second countable Baire spaces. Let X be regular and let $\mathcal{F}$ be a countable family of quasi-continuous bijections from X onto Y. Let a Blumberg set exist for f^{-1} for any $f \in \mathcal{F}$. Then $\mathcal{F}$ admits a simultaneous Blumberg set if and only if for any $f \in \mathcal{F}$ the inverse function f^{-1} is quasi - continuous.

6.3 Generalized Blumberg sets

Replacing in definition 6.1.1 the condition that $f \mid D$ is continuous by the requirement that $f \mid D$ is quasi-continuous we obtain the definition of quasi-Blumberg set. A quasi-Blumberg set D for f is said to be a full quasi-Blumberg set if $f(G \cap D)$ is dense in $f(G)$ for any G open.

By an analogous change in definition 6.1.5 we obtain the notions of upper (lower) quasi-Blumberg sets for a multifunction $F : X \to Y$.

The notion of a quasi-Blumberg set is natural because the following results are evident.

6.3.1 ([NE 10]) A multifunction $F : X \to Y$ is upper (lower) quasi-continuous if and only if any upper (lower) quasi-Blumberg set for F is its full upper (lower) quasi-Blumberg set.

6.3.2 ([NE 10]) A mapping $f : X \to Y$ is quasi-continuous if and only if any quasi-Blumberg set for f is its full quasi-Blumberg set.

The results of preceding sections may be generalized also for quasi-Blumberg sets ([NE 10]). We formulate only one of them for the single-valued case.

6.3.3 ([NE 10], see also [NL]). Let Y be a regular space. Let D be a quasi-Blumberg set for a single-valued mapping $f : X \to Y$. Then f is quasi-continuous if and only if D is a full quasi-Blumberg set for f.

Another generalization of Blumberg sets may be obtained when somewhat continuity instead of quasi-continuity is considered. (See [NL] and [NE 10].)

7. Quasi-continuity of real functions

7.1 Order continuity

A well-known notion for real functions $f : X \to R$ is the upper (lower) semi-continuity. To avoid misunderstanding with notions introduced earlier in this paper we call this notion order upper (order lower) continuity. It is defined as follows.

7.1.1 A function $f : X \to R$ is said to be order upper (order lower) continuous at $p \in X$ if for any $\varepsilon > 0$ there exists a neighborhood U of p such that $f(x) < f(p) + \varepsilon$ ($f(x) > f(p) - \varepsilon$) for any $x \in U$. It is said to be order upper (order lower) continuous if it is order upper (order lower) continuous at any $x \in X$.

The definition of order upper (order lower) quasi-continuity is natural (cf. also [EL 2] where it is called upper (lower) quasi-continuity).

7.1.2 A function $f : X \to R$ is said to be order upper (order lower) quasi-continuous at $p \in X$ if for any $\varepsilon > 0$ and any neighborhood U of p there exists a nonempty open set $G \subset U$ such that $f(x) < f(p) + \varepsilon$ ($f(x) > f(p) - \varepsilon$) for any $x \in G$. It is called order upon (order lower) quasi-continuous if it is order upper (order lower) quasi-continuous at any $x \in X$.

The notion of order upper (lower) quasi-continuity raises various natural questions, e.g. the question of the set of continuity points. Also properties of limit functions of convergent sequences of order upper (lower) quasi-continuous functions and product quasi-continuity questions have been studied. We introduce some of these results and also some references to others. We mention also how various of these results may be obtained using suitable known results for multifunctions.

Standard considerations show that the uniform convergence preserves both order upper and order lower quasi-continuity. Similarly, as in the case of quasi-continuity, one can easily find examples showing that the order upper (order lower) quasi-continuity is not preserved under the pointwise convergence. But the following is true.

7.1.3 ([EL 2]) If $\{f_n\}_{n=1}^{\infty}$ is an increasing (decreasing) sequence of order lower (order upper) quasi-continuous functions then the pointwise limit f is order lower (order upper) quasi-continuous.

As to the set of continuity points of the limit function we have

7.1.4 ([EP]) If f is a pointwise limit of a sequence of order upper (order lower) quasi-continuous functions, then the set of points where f is not lower (upper) continuous is of first category.

As to the quasi-uniform convergence one can use the example 5.1.3 to show that the limit of a sequence of order upper quasi-continuous functions need not be order lower quasi-continuous. Of course, a similar example can be given for a sequence of order lower quasi-continuous functions.

Both order upper and order lower quasi-continuity is preserved by transfinite convergence under some rather general assumptions. Namely, the following theorem holds.

7.1.4 Let X be first countable locally separable topological space. Let $\{f_\xi\}_{\xi<\omega_1}$ be a transfinite sequence of order upper (order lower) quasi-continuous real functions defined on X converging to f. Then f is order upper (order lower) quasi-continuous.

The proof of 7.1.4 is quite analogous to that one of theorem 5.2.3. Similarly as in theorem 5.2.3 the first countability may be omitted if the strong local separability is supposed.

Before giving further results we mention a relation to multifunctions.

7.1.5 ([NE 1]). Let $f : X \to R$ be a function. Let $F : X \to R$ be the multifunction defined as $F(x) = \{y : y \leq f(x)\}$. Then F is upper quasi-continuous (lower quasi-continuous) at $p \in X$ if and only if f is order upper (order lower) quasi-continuous.

Using the characterization 7.1.5 and results of section 4.1 we obtain the following results:

7.1.6 ([NE 1], see also [EL 2]). Let X be a Baire space, Y locally second countable. Let $f : X \times Y \to R$ be a multifunction such that for every $x \in X$ the section f_x is order lower quasi-continuous and for every $y \in Y$ the section f^y is both order upper and order lower quasi-continuous. Then f is order lower quasi-continuous.

7.1.7 ([NE 1], [EL 2]) Let X be a Baire space, Y locally second countable. Let $f : X \times Y \to R$ be such that f_x are order upper quasi-continuous and f^y both order upper and order lower quasi-continuous. Then f is order upper quasi-continuous.

Note that an analogous result for the order lower continuity may be obtained. A more general formulation for k-Baire spaces is also possible (cf. 4.1.5 and 4.1.6).

We leave to the reader the definitions of order upper and order lower somewhat continuity as well as the formulation of theorems on product upper and product lower somewhat continuity.

The details are given in [NE 1].

The same methods may be applied also for obtaining various other results on order upper and order lower quasi-continuity.

7.2 Quasi-continuity and Lebesgue measurability

The problem whether a quasi-continuous function $f : [0,1] \to R$ is Lebesgue measurable was solved in a negative way by S. Marcus [MC 1] who proved

7.2.1 ([MC 1]) There exists a quasi-continuous function $f : [0,1] \to R$ which is not Lebesgue measurable.

In the same paper S. Marcus proved

7.2.2 To any ordinal number $\alpha < \omega_1$ there exists a quasi-continuous function $f : [0,1] \to R$ such that f belongs to the Baire class α and does not belong to any class $\beta < \alpha$.

Evidently, the result 7.2.1 implies the following (cf. 1.2.3 and 1.2.4):

7.2.3 There exists a quasi-open set which is not Lebesgue measurable.

In this connection a question arises how it is with the strong quasi-continuity (α-continuity). Evidently, every strongly quasi-continuous function $f : [0,1] \to R$ is Lebesgue measurable since it is continuous (cf. 2.2.8). So there is no analogy to 7.2.1 for the strong quasi-continuity. But there is an analogy to 7.2.3 for strongly quasi-open (i.e. α-open) sets. In fact, the following strengthening of

7.2.3 holds.

7.2.4 ([NE 14]) There exists an α-open set which is not Lebesgue measurable.

Proof. Let B be a nowhere dense closed set in $[0,1]$ of positive Lebesgue measure. Let $Z \subset B$ be a non-measurable set. Put $A = ([0,1] - B) \cup Z$. Then, evidently, A is not Lebesgue measurable. The set A is α-open because $A^0 \supset [0,1] - B$, $\overline{A^0} = [0,1]$. Hence $(\overline{A^0})^0 = [0,1] \supset A$.

Considering multifunctions $F : [0,1] \to R$ we can adopt the following definition of measurability.

7.2.5 A multifunction $F : [0,1] \to R$ is Borel (Lebesgue) measurable if $F^-(V)$ is Borel (Lebesgue) measurable for every open set $V \subset R$.

One sees immediately (cf. 2.2.14) that also an α-continuous multifunction $F : [0,1] \to R$ is Lebesgue measurable. But the situation is different if we consider α-upper (α-lower) continuous multifunctions.

7.2.6 ([NE 14]) Let A be an α-open set which is not Lebesgue measurable (7.2.4). Define

$$F(x) = \begin{cases} \{1\} & \text{if} \quad x \in A \\ \{0,1\} & \text{if} \quad x \notin A \end{cases}, \quad G(x) = \begin{cases} \{0,1\} & \text{if} \quad x \in A \\ \{1\} & \text{if} \quad x \notin A \end{cases}.$$

Then both F and G are not Lebesgue measurable, while F is α-upper continuous and G α-lower continuous.

To conclude this section we remark that the sets of points of quasi-continuity of Lebesgue measurable functions were studied in [KT 2] where it is proved that the set of all quasi-continuity points of a Lebesgue measurable function is Lebesgue measurable. Some extensions of this results are given in [NE 14].

8. Applications of quasi-continuity

8.1 Quasi-continuity and topological properties

Various properties which are known to be preserved under continuous mappings are preserved also by quasi-continuous mappings. There are several papers which are devoted to the study of such properties. In this section we give some informations and also references to more general attitudes concerning these problems. Remark that in some of the referred papers quasi-continuity is called almost continuity while somewhat continuity is called feeble continuity (see e.g. [HC]).

Preserving separability is one of the important properties of quasi-continuity. In fact, somewhat continuity is sufficient.

8.1.1 ([FL 2]) Let X be a separable topological space and $f : X \to Y$ a somewhat continuous mapping onto Y. Then Y is separable.

So we have a corollary

8.1.2 If $f : X \to Y$ is a quasi-continuous mapping of a separable space X onto a space Y, then Y is a separable space.

The assertion 8.1.1 (and hence 8.1.2) follows from 2.1.10.

If f is a quasi-continuous bijection of a space X onto a separable space Y then X need not be separable.

The following may serve as an example.

8.1.3 Let $X = R$ with the discrete topology and $Y = R$ with the natural topology on R. Then the identity function $f : X \to Y$ is a continuous, hence quasi-continuous bijection of X onto a separable space Y, but X is not separable.

Nevertheless we have

8.1.4 Let $f : X \to Y$ be a quasi-continuous bijection with the property that for any nonempty open set $G \subset X$ we have $(f(G))^{\circ} \neq \emptyset$. Then X is separable if and only if Y is separable.

Proof. It follows immediately from the assumptions that the inverse mapping f^{-1} is somewhat continuous. So the result follows from 8.1.1.

Of course, as one can see, the condition of quasi-continuity in 8.1.4 may be replaced by somewhat continuity.

Certain types of quasi-continuous functions preserve Baire spaces ([FL 2], [NE 8], [DO 2]). We formulate such a result for k-Baire spaces omitting the proof, since it is essentially the same as for Baire spaces.

8.1.5 Let $f : X \to Y$ be a quasi-continuous mapping of a k-Baire space X onto a space Y. If for any G open, $G \neq \emptyset$, $G \subset X$ we have $(f(G))^{\circ} \neq \emptyset$, then Y is a k-Baire space.

In connection with bijections $f : X \to Y$ which are somewhat continuous and satisfy the condition $(f(G))^{\circ} \neq \emptyset$ for any nonempty open G (such bijections are called somewhat homeomorphisms since f^{-1} is also somewhat continuous) it is worth mentioning that they need not be quasi-continuous as it is sometimes noted by mistake ([SR], [HC]).

8.1.6 Let $X = Y = R$ with the natural topology. Put $f(x) = x$ if $x \neq 0$, $x \neq 1$, $f(0) = 1$, $f(1) = 0$. Then f is a somewhat homeomorphism but it is not quasi-continuous.

In spite of the fact that somewhat homeomorphisms need not be quasi-continuous we have the following result generalizing 8.1.5.

8.1.7 If $f : X \to Y$ is a somewhat homeomorphism then X is a k-Baire space if and only if Y is a k-Baire space.

Proof. Let X be a k-Baire space. Let $\{G_t : t \in T\}$ be a collection of cardinality less than k of open dense sets in Y. We prove that the sets $Z_t = (f^{-1}(G_t))^\circ$ are dense in X. Let $t \in T$, $p \in X$, and U an arbitrary neighborhood of p. Then there exists a nonempty open set $V \subset f(U)$. The set $V \cap G_t$ is nonempty and since f is somewhat continuous we have a nonempty open set W such that

$$W \subset f^{-1}(V \cap G_t) \subset f^{-1}(f(U)) = U.$$

Thus in any neighborhood U of p there is a point belonging to W and hence to Z_t. So the density of Z_t is proved. Since X is a k-Baire space, $\bigcap_{t\in T} Z_t$ is dense in X. The somewhat continuity of f implies that $f(\bigcap_{t\in T} Z_t)$ is dense in Y. But

$$\bigcap_{t\in T} G_t \supset \bigcap_{t\in T} f(Z_t) \supset f(\bigcap_{t\in T} Z_t).$$

Thus $\bigcap_{t\in T} G_t$ is dense in Y. So Y is a k-Baire space. The "only if" part follows from the fact that f^{-1} is also somewhat continuous and somewhat open.

As a corollary we have

8.1.8 ([NE 8], [FL 2], [SR], [HC]) Let $f : X \to Y$ be a somewhat homeomorphism. Then X is a Baire space if and only if Y is a Baire space.

Various other applications of quasi-continuity to topological questions can be found in the papers mentioned above ([FL 2], [HC], [NE 8], [SR]).

There are also many topological applications of mappings closely related to quasi-continuity. In this connection we refer the reader to [NO 1], [NO 2], [RV], where the strong quasi-continuity in connection with preservation of connected spaces is studied. The notion of quasi-homeomorphism (it is also called semi-homeomorphism) was introduced and studied by S.G. Crossley and S.K. Hildebrand ([CH 1], [CH 2]). They investigated preservation of various topological properties under mappings closely related to quasi-continuous mappings. Similar questions are studied also in [BI], [NE 9], [PT 7], [NO 3]. Certain relations between quasi-continuity of multi-functions on Baire spaces are investigated in [EW 3].

8.2 Quasi-continuity and differentiability

In this section let E denote an open n-dimensional cube in R_n. Let $f : E \to R$ ($f(x) = f(x_1, \ldots, x_n)$). A well known theorem of mathematical analysis states that the continuity of (finite) partial derivatives of f is sufficient for the differentiability of f. On the other hand there are well known examples showing that the existence of finite partial derivatives does not imply the differentiability. In this connection an interesting result has been achieved by S. Marcus. The simple proof of the mentioned result, which we give for the convenience of the reader, is a simple application of two theorems on quasi-continuity.

8.2.1 ([MC 1]) If $f : E \to R$ possesses finite partial derivatives $\frac{\partial f}{\partial x_i}$ ($i = 1, 2, \ldots, n$) on E, then the set of those points where f is not differentiable is of first category.

Proof. For $i = 1, 2, \ldots, n$ we have $\frac{\partial f}{\partial x_i} = lim_{m \to \infty} g_{m,i}$, where

$$g_{m,i}(x) = m(f(x_1, \ldots, x_i + \frac{1}{m}, \ldots, x_n) - f(x_1, \ldots, x_n)).$$

The functions $g_{m,i}$ are separately continuous, hence by 4.1.9 they are quasi-continuous. So $\frac{\partial f}{\partial x_i}$ is the limit of a sequence of quasi-continuous functions. By 5.1.1 the set of discontinuity points of each partial derivative $\frac{\partial f}{\partial x_i}$ is of first category. Thus with the exception of a set of first category all the partial derivatives are continuous, hence f is differentiable with the exception of a set of first category.

It should be noted that a theorem more general than 8.2.1 was obtained in [WL] by a different method. Namely the existence of partial derivatives on a dense G_δ subset of E implies the differentiability on a dense G_δ subset of E.

Further results of S. Marcus using quasi-continuity concern the problem of interchange of partial derivatives.

On the basis of quasi-continuity we present a theorem covering and unifying two theorems of S. Marcus concerning the interchange of partial derivatives. The result is due to A. Neubrunnová (unpublished).

8.2.2 ([SA 2]) Let $I \subset (-\infty, \infty)$ be an interval. A function $f : I \to R$ is said to be L-continuous at $p \in I$ if for any $\varepsilon > 0$, $\delta > 0$ the set $\{x : x \in (p-\delta, p+\delta) \cap I; \mid f(x) - f(p) \mid < \varepsilon\}$ is of positive Lebesgue measure. If f is L-continuous at any $x \in I$ it is called L-continuous (on I).

The following simple lemma will be useful.

8.2.3 ([NA 2]) Let $f : I \to R$ be almost everywhere (in the sense of the Lebesgue measure) continuous and L-continuous. Then it is quasi-continuous.

The next is the mentioned result about the interchange of the order of differentiation.

8.2.4 Let $f : E \to R$ and $p > 1$ be an integer. Suppose that the partial derivatives of order p are almost everywhere separately continuous and separately

L-continuous. Moreover, let the difference of any two partial derivatives of the order p differing only in the order of differentiation be L-continuous. Then the derivatives of the order less than or equal to p do not depend on the order of differentiation.

Proof. For the partial derivatives of order less than p our assertion follows from the classical result. So, let g, h be any two of partial derivatives of order p differing only in the order of differentiation. From the almost everywhere continuity and L-continuity of g, h in each variable separately it follows (8.2.3) that g, h are separately quasi-continuous; hence, by 4.1.9, they are quasi-continuous and therefore (3.2.1) continuous on a dense set $Z \subset E$. Thus by the classical theorem on interchange of the order of differentiation $g = h$ on Z. But, by assumption, the function $\ell = g - h$ is almost everywhere continuous and L-continuous in each variable separately. So, by 8.2.3, it is quasi-continuous in each variable and, by 4.1.9, it is quasi-continuous as a function of n variables. So ℓ is a quasi-continuous function which is equal to zero on a dense subset of E. Then, evidently, $\ell(x) = 0$ for any $x \in E$ and the theorem is proved.

Now, if we have a function $f : I \to R$ which is a derivative or which is approximately continuous then ([NA 2]) it is L-continuous. So we obtain as corollaries of 8.2.4 two results of S. Marcus.

8.2.5 ([MC 2]) Let $f : E \to R$ and let $p > 1$ be an integer. Let the mixed partial derivatives of order p be - as functions of any of the variables - almost everywhere continuous derivatives. Then the mixed partial derivatives of order less than or equal to p on E do not depend on the order of differentiation.

8.2.6 ([MC 2]) Let $f : E \to R$ and let $p > 1$ be an integer. If the mixed partial derivatives of order p are in each variable separately almost everywhere continuous and approximately continuous then the mixed partial derivatives of order less than or equal to p on E do not depend on the order of differentiation.

8.3 Quasi-continuity in measure and probability theory

Here we show some applications of quasi-continuity to measure theory and to probability theory.

A well known and important theorem in measure theory is the so-called Jegoroff's theorem ([HA]) asserting that the pointwise convergence of a sequence of measurable functions on a set of finite measure is almost uniform.

Quasi-continuity may be used to prove a generalized version of Jegoroff's theorem.

8.3.1 Let T be a topological space, $t_0 \in T$. Let $\{f^t : t \in T\}$ be a family of measurable real functions defined on X, where (X, S, μ) is a measure space. Then $\{f^t : t \in T\}$ is said to be almost uniformly convergent, as t tends to t_0, to a function φ defined on X if to any $\varepsilon > 0$ there exists a set $E \in S$ with $\mu(E) < \varepsilon$ such that $\{f^t : t \in T\}$ converges uniformly to φ on $X - E$.

While in the case of a discrete parameter t the pointwise convergence implies the almost uniform convergence ([HA]), the situation in case of a parameter running over a general space T is different. It was proved in [TO] (see also [WE] and [WA]), that in the general case the pointwise convergence need not imply the almost uniform convergence. Sufficient conditions are known for the validity of Jegoroff's theorem. Usually the continuity or Borel measurability of the function $f^t(x)$ as a function of variable t for each fixed $x \in X$ is supposed. (These conditions concern the case of a real parameter t - see e.g. [TO], [WE], [WA]). We give a condition where the quasi-continuity in the variable t is assumed.

8.3.2 ([NE 12]) Let (X, S, μ) be a totally finite measure space, T a separable topological space satisfying the first countability axiom and $t_0 \in T$. Let $\{f^t : t \in T\}$ be a family of real measurable functions defined on X such that $\lim_{t \to t_0} f^t(x) = \varphi(x)$ for any $x \in X$. Let for any $x \in X$ $f^t(x)$ be quasi-continuous on T. Then $\{f^t : t \in T\}$ converges to φ almost uniformly.

We refer the reader to [NE 12,13] for the proof of 8.3.2 as well as for some related questions.

Perhaps it is worth mentioning that the collection $\{f^t : t \in T\}$ considered above in case when (X, S, μ) is a probability space (usually $T = R$ is supposed) may be viewed as a stochastic process (see e.g. [YE]). Among the stochastic processes the separable ones are important. Recall that $\{f^t : t \in T\}$ is called a continuous stochastic process if for any $x \in X$ the function $f^t(x)$ is continuous on T. It is called separable ([YE], p. 26) if there exists a countable set $C \subset T$ such that for each closed interval $I \subset (-\infty, \infty)$ and every open set $G \subset T$ we have

$$\{x : f^t(x) \in I \text{ for every } t \in G\} = \{x : f^t(x) \in I \text{ for every } t \in C \cap G\}.$$

As is well known ([YE]) any continuous stochastic process is separable. Replacing the condition of continuity by that of quasi-continuity we obtain in a natural way the definition of a quasi-continuous process. For these types of processes we have

8.3.3 ([NE 13]) Any quasi-continuous stochastic process is separable.

We refer to [NE 13] for the proof of 8.3.3 as well as for some further relations of quasi-continuity to stochastic processes.

Note that quasi-continuity was used also in connection with measurability of certain multifunctions ([TM]).

Also quasi-continuous selections of multifunctions were studied [MD 2]. For an attempt of using quasi-continuity in some optimization problems related to mathematical programming see [NR 1,2].

Concluding remarks

There is a number of results on quasi-continuity which do not belong directly to any of the previous sections. Some of them will be mentioned here.

A connection between derivatives and quasi-continuous real functions of a real variable was discussed in [MR]. A quasi-continuous function need not be a derivative and a derivative need not be quasi-continuous. A derivative which is Riemann integrable on every segment is quasi-continuous ([MR]). Similar results may be found in [NA 2], [SA 2], [PP].

In the case of real functions of a real variable also the linear space spanned by cliquish functions was studied. In this connection it was proved ([GR]) that any cliquish function is the sum of four quasi-continuous functions.

The stationary and determining sets ([BR]) for quasi-continuous functions were studied in [DO 1,2].

The topological structure of the set $Q(X)$ of all bounded real quasi-continuous functions defined on a topological space X in the normed space $M(X)$ of bounded real functions with the sup-norm was studied in [SA 2]. The set $Q(X)$ was proved to be perfect in $M(X)$ and in non-trivial cases it is nowhere dense in $M(X)$. Some closely related results can be found in [SA 2], [SA 4].

The lattice generated by $Q(X)$ was studied in [GN].

Quasi-continuous selections of multifunctions were studied in [MD 2].

In [PO 1] a decomposition of quasi-continuity into two types of generalized continuities was investigated. This is an analogue to a decomposition of continuity studied in [LE 1].

Some of the results on quasi-continuity mentioned in this paper may be formulated in a more general setting. We have in mind abstract approaches similar to those given e.g. in [PW], [TB], [TH], [WN], [MD 4].

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