On Rates of Convergence of Certain Sequences

W.A. Beyer, Theoretical Division, MS B284
Los Alamos National Laboratory
Los Alamos, NM 87545

and

*B.R. Ebanks, Department of Mathematics
University of Louisville
Louisville, KY 40292

In the study of periodic orbits of maps of the interval under iteration, it has been observed by many people that there are certain behaviors which are "universal" for large classes of one-parameter families of maps. To illustrate, let $f:[0,1] \to [0,\infty)$ be such that:

(a) f is nondecreasing on $[0,1/2]$, nonincreasing on $[1/2,1]$.

(b) $f(0) = f(1) = 0$, $f(1/2) = \max\{f(x) \mid x \in [0,1]\}$.

Such an f will be called unimodal (although some authors reserve that term to imply a single maximum). Then define a one-parameter family $\{F_\lambda\}$ of transformations of the unit interval by

$$F_\lambda(x) = \lambda f(x), \qquad x \in [0,1], \tag{1}$$

where λ is restricted to a domain for which $F_\lambda:[0,1] \to [0,1]$. Necessary and sufficient conditions on f for the "universal" properties to hold are not known,

*Presenter

but f is generally assumed to be continuous on $[0,1]$ and piecewise differentiable, or at least differentiable in a neighborhood of 1/2.

We define the iterates of F_λ as follows:

$$F_\lambda^0(x) = x,\ F_\lambda^1(x) = F_\lambda(x),\ F_\lambda^n(x) = F_\lambda(F_\lambda^{n-1}(x)), n = 2,3,\ldots$$

For each positive integer m, we are interested in solutions λ of the equation

$$F_\lambda^m(1/2) = 1/2. \qquad (2)$$

For various subclasses of unimodal functions, including trapezoid functions [9], parabolic functions [6] and others [1], it has been shown that each solution λ of (2) corresponds to a unique cycle (periodic orbit) of length m, one of whose points is 1/2. These cycles are distinguished by their patterns[1], words (of length $m-1$ for an m-cycle) on the alphabet $\{R,L\}$. R (resp. L) indicates that an iterate lies to the right (resp. left) of 1/2. For example, the (unique) 3-cycle has the pattern RL, meaning that $F_\lambda(1/2) > 1/2$, $F_\lambda^2(1/2) < 1/2$, $F_\lambda^3(1/2) = 1/2$. The countable set of all cycles (for all m) can be uniquely ordered on increasing λ. (Of course, the values of λ depend on f.) The resulting sequence of patterns and their ordering [10,5] is universal for a large class of unimodal functions, including the subclasses mentioned above. This sequence of patterns will be called the MSS sequence, as it was first observed to be universal by Metropolis, Stein, and Stein [10].

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[1] also called finite shift-maximal sequences.

The MSS sequence exhibits a phenomenon known as <u>period doubling</u>, described as follows. Associated with any pattern $\underline{P}$ of length k there is a sequence of patterns $H_n(\underline{P})$ (see below), called harmonics of $\underline{P}$, of lengths $k2^n (n = 0,1,\ldots)$, each of which corresponds to a solution λ_n of (2) with $m = k2^n$. Each sequence $\{\lambda_n\}$ (for each k) converges; moreover, Feigenbaum [7] observed that the ratio

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \tag{3}$$

also seems to converge to a number $\delta(d)$ which is the same for all functions in some allowed class which have the same degree d in a neighborhood of the maximum $(x = 1/2)$. This conjecture of Feigenbaum's was proved (for functions in a suitable class) when $d = 1 + \varepsilon$ for small positive ε by Collet <u>et</u>. <u>al</u>. [4,3] and when $d = 2$ by Lanford [8]. Convergence of (3) is described as geometric convergence of $\{\lambda_n\}$.

Somewhat surprisingly, however, it was found by Beyer and Stein [2] that the rate of convergence of $\{\lambda_n\}$ is different for functions which are flat in a neighborhood of 1/2. Specifically, for trapezoid functions (defined below), they show that

$$\lim_{n\to\infty} \frac{\log\left(\dfrac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}}\right)}{\log\left(\dfrac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}\right)} = 2 \tag{4}$$

while (3) diverges. It has been shown recently by Beyer, Ebanks, and Qualls [11] that (4) is equivalent to quadratic (i.e., order 2) convergence of $\{\lambda_n\}$.

But there is an even "more universal" convergence phenomenon that has been observed in numerical experiments. It has to do with the rate of convergence of certain projections $g_{H_n(\underline{P})}(\lambda)$ (defined below).

Given $e \varepsilon (0,1/2)$, consider the one-parameter family $\{F_\lambda\}$ of <u>trapezoidal</u> <u>maps</u> on $[0,1]$ defined by (1) with

$$f(x) = \begin{cases} e^{-1}x & , \quad 0 \leq x \leq e \\ 1 & , \quad e \leq x \leq 1-e \\ e^{-1}(1-x) & , \quad 1 - e \leq x \leq 1, \end{cases}$$

where λ ranges in $(e,1]$. This is a one-parameter family in the sense that e is regarded as fixed in each discussion.

Each number y in $[0,\lambda)$ has two inverses under F_λ, denoted

$$F^{-1}_{\lambda,R}(y): = 1 - \lambda^{-1}ey, \quad F^{-1}_{\lambda,L}(y): = \lambda^{-1}ey. \tag{5}$$

We extend $F^{-1}_{\lambda,L}$ and $F^{-1}_{\lambda,R}$ to $[0,1]$ by requiring (5) to hold also for $y\varepsilon[\lambda,1]$.

As in [1], we define for each pattern $\underline{P} = P_1P_2 \ldots P_n$ the function

$$G_\lambda(\underline{P},y): = F^{-1}_{\lambda,P_1}(F^{-1}_{\lambda,P_2}(\ldots F^{-1}_{\lambda,P_n}(y)\ldots)),$$

and the <u>inverse</u> <u>sequence</u> <u>function</u> $g_{\underline{P}}$ by

$$g_{\underline{P}}(\lambda): = G_\lambda(\underline{P},1/2).$$

One reason for studying the inverse sequence function is that if $\lambda_{\underline{P}}$ corresponds to a pattern $\underline{P}$, then $\lambda_{\underline{P}}$ is a fixed point of $g_{\underline{P}}$, i.e. $g_{\underline{P}}(\lambda_{\underline{P}}) = \lambda_{\underline{P}}$. This is the point of view taken by those studying Feigenbaum's convergence. Rather than studying sequences of fixed points, however, we shall be interested in projections of those sequences. That is, we shall examine a sequence of values of $g_{\underline{P}}(\lambda)$ for fixed λ as $\underline{P}$ runs through a sequence of patterns called harmonics.

The harmonics of a pattern $\underline{P}$ are defined as follows. If $\underline{P} = P_1P_2 \dots P_n$, then the <u>first</u> <u>harmonic</u> of $\underline{P}$, denoted $H_1(\underline{P})$, is the pattern

$$H_1(\underline{P}) = P_1P_2 \dots P_n \, P_{n+1} \, P_1P_2 \dots P_n,$$

where P_{n+1} is chosen so that the number of R's in $\underline{P}$ and the number of R's in $H_1(\underline{P})$ will have opposite parity. Higher harmonics are defined inductively:

$$H_n(\underline{P}) = H_1(H_{n-1}(\underline{P})), \quad n = 2,3,\dots,$$

and the <u>zero-th harmonic</u> of $\underline{P}$ is $\underline{P}$ itself:

$$H_0(\underline{P}) = \underline{P}.$$

As an example, the sequence $\{H_n(R)\}$ (starting at n = 0) is

$$H_0(R) = R, \; H_1(R) = RLR, \; H_2(R) = RLR^3LR, \dots.$$

The sequences of <u>projections</u> which are the main subjects of this paper are $\{g_{H_n(\underline{P})}(\lambda)\}_n$ for fixed λ and fixed patterns $\underline{P}$.

<u>Theorem</u>. Let $\underline{P}$ be a pattern, define $\zeta_n(t) := g_{H_n(\underline{P})}(\lambda)$ for all $n(=0,1,2,\dots)$, and let $\lambda \in (e,1]$ and $t = e/\lambda$. Then $\{\zeta_n\}$ converges on $[e,1)$ to an analytic function w, and

$$\lim_{n\to\infty} \frac{\zeta_{n+1}(t) - \zeta_n(t)}{[\zeta_n(t) - \zeta_{n-1}(t)]^2} = \frac{t}{1/2 - t\,w\,(t)}, \quad t \in [e,1).$$

In particular, $\{\zeta_n(t)\}$ converges quadratically for each t with $2tw(t) \neq 1$.

We have numerical evidence that results similar to those contained herein for trapezoidal maps hold also for parabolic and other unimodal maps. In fact, we also have partial results towards a proof for parabolas.

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