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LEBESGUE POINTS OF FRACTIONAL INTEGRALS

1. Introduction.

Fractional integrals. Let $f \in L(a,b)$ and $\operatorname{re} c > 0$. We define a cth integral of f to be the function $I^c f$ given by

$$I^c f(x) = (I^c f)(x) = \int_a^x \frac{(x-t)^{c-1}}{\Gamma(c)} f(t)\, dt; \tag{1}$$

this is the Riemann-Liouville fractional integral of f of order c.

Much work has been done on integrability- and continuity-type properties of $I^c f$ for various kinds of function f. The main landmark in this is the work of Hardy and Littlewood [1], and it is sometimes thought that they exhausted this field. However, they did not consider Lebesgue points of $I^c f$, and this is the subject of this paper. The main interest is in $0 < c < 1$, to which we confine attention.

A fundamental property is that

$I^c f(x)$ exists for almost all $x \in (a,b)$ and is integrable thereon. (2)

This follows from $I^c f$ being a convolution of integrable functions. However, much more may be true; for instance, considering $c = 1$,

$$I^1 f(x) = \int_a^x f(t)\, dt$$

exists for all $x \in [a,b]$ and is absolutely continuous thereon.

This suggests, and Hardy and Littlewood's many theorems in [1] support, the view that the continuity-type properties of $I^c f$ improve as c increases. For instance, their Theorem 12 shows that under certain conditions $I^c f$ belongs to a Lipschitz class which contracts as c increases. Indeed, the essential message of that theorem amounts, in brief, to:

<u>If</u> $f \in L^p$ <u>and</u> $\frac{1}{p} < c < 1$, <u>then</u> $I^c f \in \mathrm{Lip}(c - \frac{1}{p})$.

Many of their results in [1], like this one, were for $f \in L^p$ with $p > 1$; and they showed that most of them were false for $p = 1$. In this present paper all results are concerned with $f \in L^1$.

2. Lebesgue points.

<u>Lebesgue points</u> of $g \in L$ are points ξ such that both

$$\frac{1}{h}\int_0^h |g(\xi \pm s) - g(\xi)|\, ds \to 0 \quad \text{as} \quad h \to 0+ \, ; \tag{3}$$

a continuity-type property, weaker than continuity. We shall abbreviate "Lebesgue point" to "L-point".

By a fundamental theorem, for $g \in L$ almost all points are L-points. Consequently for $f \in L$

<u>almost all points are L-points of $I^c f$</u>,

by (2). But (2) also gives that

<u>almost all points are existence-points of $I^c f$</u>.

My theme in this paper is <u>broadly</u> that L-points and existence-points of $I^c f$ are the same points.

Every L-point is an existence-point, merely by the definition (3); so my task is to prove the converse, that every existence-point is a L-point. The converse is not quite true in this simple form; the final form will be seen in Theorem 3.

Throughout the paper the same things can be said with "L-point" replaced by "point of approximate continuity", since Lebesgue points are necessarily points of approximate continuity.

3. Left Lebesgue points.

<u>Theorem 1</u>. <u>If</u> $0 < c < 1$, $f \in L(a,b)$, $\xi \in (a,b]$ <u>and</u> $I^c f(\xi)$ <u>exists</u>, <u>then</u> ξ <u>is a left L-point of</u> $I^c f$; <u>that is</u>,

$$\frac{1}{h}\int_0^h |I^c f(\xi - s) - I^c f(\xi)|\, ds \to 0 \quad \underline{\text{as}} \quad h \to 0+.$$

Proof. Let $0 < h < \eta < \frac{1}{2}(\xi - a)$. Since by (2) $I^c f$ exists almost everywhere in (a,b) we have, for almost all $s \in (0,h)$,

$$\Gamma(c)\{I^c f(\xi) - I^c f(\xi - s)\} = \left(\int_a^{a+s} + \int_{a+s}^{\xi-\eta} + \int_{\xi-\eta}^{\xi} \right) (\xi - t)^{c-1} f(t)\, dt$$

$$- \left(\int_{a+s}^{\xi-\eta} + \int_{\xi-\eta}^{\xi} \right) (\xi - u)^{c-1} f(u-s)\, du,$$

so that

$$L = \frac{\Gamma(c)}{h} \int_0^h |I^c f(\xi) - I^c f(\xi - s)|\, ds$$

$$\leq \frac{1}{h}\int_0^h ds \int_a^{a+s} (\xi-t)^{c-1}|f(t)|dt + \frac{1}{h}\int_0^h ds \int_{a+s}^{\xi-\eta} (\xi-t)^{c-1}|f(t) - f(t-s)|dt$$

$$+ \frac{1}{h}\int_0^h ds \int_{\xi-\eta}^{\xi} (\xi-t)^{c-1}|f(t)|dt + \frac{1}{h}\int_0^h ds \int_{\xi-\eta}^{\xi} (\xi-t)^{c-1}|f(t-s)|dt$$

$= L_1 + L_2 + L_3 + L_4$, say.

We aim to make as much of this as possible independent of h and s.

$$L_1 + L_3 \leq \int_a^{a+\eta} (\xi-t)^{c-1}|f(t)|dt + \int_{\xi-\eta}^{\xi} (\xi-t)^{c-1}|f(t)|dt = M_1 + M_3,$$

$$L_2 \leq \eta^{c-1}\ \frac{1}{h}\int_0^h ds \int_{a+s}^{\xi-\eta} |f(t) - f(t-s)|dt$$

$$\leq \eta^{c-1} \sup_{0<s<h} \int_{a+s}^{\xi-\eta} |f(t) - f(t-s)|dt = M_2,$$

$$L_4 = \frac{1}{h}\int_0^h ds \int_s^{s+\eta} (v-s)^{c-1}|f(\xi-v)|dv \qquad \text{by } t-s = \xi - v,$$

$$\leq \frac{1}{h}\int_0^h ds \int_s^{2\eta} (v-s)^{c-1}|f(\xi-v)|dv$$

$$\le \frac{1}{h}\int_0^{2h} ds \int_s^{2h} (v-s)^{c-1}|f(\xi-v)|dv + \frac{1}{h}\int_0^{h} ds \int_{2h}^{2\eta} (v-s)^{c-1}|f(\xi-v)|dv$$

$$\le \frac{2}{2h}\int_0^{2h} |f(\xi-v)|dv \int_0^{v} (v-s)^{c-1}\, ds + \frac{1}{h}\int_0^{h} ds \int_{2h}^{2\eta} (\tfrac{1}{2}\, v)^{c-1}\, |f(\xi-v)|dv$$

$$\le 2\int_0^{2h} \frac{|f(\xi-v)|}{v}\,\frac{v^c}{c}\, dv + \frac{2}{2^c}\int_{2h}^{2\eta} v^{c-1}|f(\xi-v)|dv$$

$$\le \frac{2}{c}\left(\int_0^{2h} + \int_{2h}^{2\eta}\right)v^{c-1}|f(\xi-v)|dv = \frac{2}{c}\int_{\xi-2\eta}^{\xi} (\xi-t)^{c-1}|f(t)|dt = M_4.$$

Now M_1, M_3 and M_4 are independent of h; but they all tend to zero with η because $I^c f(\xi)$ exists. Thus given $\varepsilon > 0$, η can be chosen small enough to make all three less than $\frac{1}{4}\varepsilon$. With η so fixed, $M_2 \to 0$ as $h \to 0$ by continuity-in-L^1-norm of f. So, given $\varepsilon > 0$ there is $\delta > 0$ such that

$$L \le M_1 + M_2 + M_3 + M_4 < \varepsilon \quad \text{whenever} \quad 0 < h < \delta,$$

as required.

4. Right Lebesgue points.

It is evident from (1) that existence of $I^c f(\xi)$ exercises no control over the values of $f(x)$ or of $I^c f(x)$ for $x > \xi$. So no analogue of Theorem 1 for right L-points of $I^c f$ can be expected without some extra hypothesis. This explains the need for (4) in the following theorem.

Theorem 2. *If* $0 < c < 1$, $f \in L(a,b)$, $\xi \in [a,b)$, $I^c f(\xi)$ *exists and*

$$\frac{1}{h}\int_0^{h} |f(\xi+t)|dt = o(h^{-c}) \quad \text{as} \quad h \to 0+, \tag{4}$$

then ξ *is right L-point of* $I^c f$; *that is,*

$$\frac{1}{h}\int_0^{h} |I^c f(\xi+s) - I^c f(\xi)|ds \to 0 \quad \text{as} \quad h \to 0+.$$

Proof. Since $I^c f \in L(a,b)$ by (2) and since $I^c f(\xi)$ exists, the expression

$$R = \frac{\Gamma(c)}{h}\int_0^h |I^c f(\xi+s) - I^c f(\xi)|\,ds$$

has meaning for $0 < h < b-\xi$.

(i) Since $I^c f(a) = 0$ by (1), $\xi = a$ is possible. In that case

$$R = \frac{1}{h}\int_0^h \left| \int_a^{a+s} (a+s-t)^{c-1} f(t)\,dt \right| ds \le \frac{1}{h}\int_0^h ds \int_0^s (s-u)^{c-1}|f(a+u)|\,du$$

$$= \frac{1}{h}\int_0^h |f(a+u)|\,du \int_u^h (s-u)^{c-1}\,ds \le \frac{h^c}{c}\,\frac{1}{h}\int_0^h |f(a+u)|\,du \to 0$$

as $h \to 0+$, by (4); thus a is a right L-point of f, as required.

(ii) Suppose that $a < \xi < b$ and $0 < h < \eta < \min\{b-\xi, \xi-a\}$. For almost all $s \in (0,h)$,

$$\Gamma(c)\{I^c f(\xi+s) - I^c f(\xi)\} = \left(\int_{a-s}^{a} + \int_a^{\xi-\eta} + \int_{\xi-\eta}^{\xi} \right) (\xi-u)^{c-1} f(u+s)\,du$$

$$- \left(\int_a^{\xi-\eta} + \int_{\xi-\eta}^{\xi} \right) (\xi-t)^{c-1} f(t)\,dt;$$

and so R is no greater than

$$\frac{1}{h}\int_0^h ds \int_a^{a+s} (\xi+s-t)^{c-1}|f(t)|\,dt + \frac{1}{h}\int_0^h ds \int_a^{\xi-\eta} (\xi-t)^{c-1}|f(t+s) - f(t)|\,dt$$

$$+ \frac{1}{h}\int_0^h ds \int_{\xi-\eta}^{\xi} (\xi-u)^{c-1}|f(u+s)|\,du + \frac{1}{h}\int_0^h ds \int_{\xi-\eta}^{\xi} (\xi-t)^{c-1}|f(t)|\,dt$$

$$= R_1 + R_2 + R_3 + R_4, \quad \text{say.}$$

As before we make as much of this as possible independent of h and s.

$$R_1 + R_4 \le \int_a^{a+\eta} (\xi-t)^{c-1}|f(t)|\,dt + \int_{\xi-\eta}^{\xi} (\xi-t)^{c-1}|f(t)|\,dt = S_1 + S_4,$$

$$R_2 \leq \eta^{c-1} \frac{1}{h} \int_0^h ds \int_a^{\xi-\eta} |f(t+s) - f(t)| dt$$

$$\leq \eta^{c-1} \sup_{0<s<h} \int_a^{\xi-s} |f(t+s) - f(t)| dt = S_2,$$

$$R_3 = \frac{1}{h} \int_0^h ds \int_{s-\eta}^{s} (s-v)^{c-1} |f(\xi+v)| dv \qquad \text{by} \quad u + s = \xi + v,$$

$$= \frac{1}{h} \left[\int_0^h ds \int_{s-\eta}^{0} dv + \int_0^h ds \int_0^s dv \right] (s-v)^{c-1} |f(\xi+v)|$$

$$\leq \frac{1}{h} \int_0^h ds \int_{-\eta}^{0} (-v)^{c-1} |f(\xi+v)| dv + \frac{1}{h} \int_0^h |f(\xi+v)| dv \int_v^h (s-v)^{c-1} ds$$

$$= \int_{\xi-\eta}^{\xi} (\xi-t)^{c-1} |f(t)| dt + \frac{1}{h} \int_0^h |f(\xi+v)| \frac{(h-v)^c}{c} dv$$

$$\leq S_4 + \frac{h^c}{c} \frac{1}{h} \int_0^h |f(\xi+v)| dv = S_4 + S_3, \quad \text{say.}$$

Now S_1 and S_4 are independent of h; and they can be made less than $\frac{1}{5}\varepsilon$ by choosing η sufficiently small. With η so fixed, $S_2 \to 0$ as $h \to 0$ by continuity-in-L^1-norm of f; and $S_3 \to 0$ as $h \to 0$ by (4). Thus, given $\varepsilon > 0$ there is $\delta > 0$ such that

$$R \leq S_1 + S_2 + S_3 + 2S_4 < \varepsilon \quad \text{whenever} \quad 0 < h < \delta,$$

as required; this completes the proof of Theorem 2.

<u>Remarks</u>. Hypothesis (4) of Theorem 2 cannot be relaxed by replacing o by O. For the function

$$f(x) = 0 \quad \text{for} \quad x \leq \xi, \qquad f(x) = (x-\xi)^{-c} \quad \text{for} \quad x > \xi$$

satisfies all the hypotheses except (4), and

$$\frac{1}{h} \int_0^h |f(\xi+t)| dt = \frac{h^{-c}}{1-c} = O(h^{-c}).$$

But ξ is not a right L-point of $I^c f$, because $I^c f$ has a simple discontinuity on the right at ξ; for if $x > \xi$

$$\Gamma(c)I^c f(x) = \int_\xi^x (x-t)^{c-1}(t-\xi)^{-c}dt = \Gamma(c)\Gamma(1-c),$$

and so as $x \to \xi+$

$$I^c f(x) \to \Gamma(1-c) \neq 0 = I^c f(\xi).$$

This example is also significant in another way. If it were true that for all integrable f <u>all</u> points were L-points of $I^c f$, Theorems 1 and 2 would be relatively pointless. But the example shows that not all points need be L-points of $I^c f$.

5. <u>Left-handed fractional integrals</u>.

For $f \in L(a,b)$ and $\mathrm{re}\ c > 0$, define $J^c f$ by

$$J^c f(x) = (J^c f)(x) = \int_x^b \frac{(s-x)^{c-1}}{\Gamma(c)} f(s)\ ds. \tag{5}$$

Writing $g(t) = f(a+b-t)$, the substitutions $s = a + b - t$ and $x = a + b - y$ show that

$$\int_x^b (s-x)^{c-1} f(s)ds = \int_a^y (y-t)^{c-1} g(t)dt,$$

and hence that

$$J^c f(x) = I^c g(y), \tag{6}$$

either side existing whenever the other does. This indicates the well-known fact that J^c has properties like those of I^c.

We need some assorted lemmas involving properties like (4).

__Lemma 1__. __If__ $0 < c < 1$, $f \in L(a,b)$, $\xi \in (a,b]$ __and__ $I^c f(\xi)$ __exists__, __then__

$$\frac{1}{h}\int_0^h |f(\xi-t)|dt = o(h^{-c}) \qquad \underline{\text{as}} \qquad h \to 0+.$$

__Proof__. $\Gamma(c)I^c f(\xi) = \int_a^{\xi} (\xi-t)^{c-1} f(t)dt = \int_0^{\xi-a} u^{c-1} f(\xi-u)du,$

so by hypothesis $u^{c-1}|f(\xi-u)|$ is integrable on $0 < u < \xi-a$.

$$h^c \frac{1}{h}\int_0^h |f(\xi-u)|du = \int_0^h h^{c-1}|f(\xi-u)|du \le \int_0^h u^{c-1}|f(\xi-u)|du;$$

this tends to 0 as $h \to 0+$, by the integrability just proved.

__Lemma 2__. __If__ $0 < c < 1$, $f \in L(a,b)$, $\xi \in [a,b)$ and $J^c f(\xi)$ __exists__, __then__

$$\frac{1}{h}\int_0^h |f(\xi+t)|dt = o(h^{-c}) \qquad \underline{\text{as}} \qquad h \to 0+.$$

__Proof__. Let $g(t) = f(a+b-t)$ and $\eta = a + b - \xi \in (a,b]$. Then $g \in L(a,b)$, and $I^c g(\eta) = J^c f(\xi)$ exists by (6), so by Lemma 1

$$\frac{1}{h}\int_0^h |f(\xi+t)|dt = \frac{1}{h}\int_0^h |g(\eta-t)|dt = o(h^{-c}).$$

__Lemma 3__. __If__ $0 < c < 1$, $f \in L(a,b)$, $\xi \in [a,b)$ __and__ $J^c f(\xi)$ __exists__, __then__ ξ __is a right L-point of__ $J^c f$. (Compare Theorem 1.)

__Proof__. Let $g(t) = f(a+b-t)$, $\eta = a+b-\xi$ and $0 < h < b-\xi$; then

$$\frac{1}{h}\int_0^h |J^c f(\xi+s) - J^c f(\xi)|ds = \frac{1}{h}\int_0^h |I^c g(\eta-s) - I^c g(\eta)|ds$$

by (6). Since $g \in L(a,b)$, $\eta \in (a,b]$ and $I^c g(\eta)$ exists, η is a left L-point of $I^c g$ by Theorem 1. So the above expressions tend to 0 as $h \to 0+$, whence $J^c f$ has a right L-point at ξ.

Lemma 4. If $0 < c < 1$, $f \in L(a,b)$, $\xi \in (a,b]$, $J^c f(\xi)$ exists and

$$\frac{1}{h}\int_0^h |f(\xi-t)|dt = o(h^{-c}) \qquad \text{as} \quad h \to 0+,$$

then ξ is a left L-point of $J^c f$. (Compare Theorem 2.)

Proof. Let $g(t) = f(a+b-t)$, $\eta = a+b-\xi$ and $0 < h < \xi-a$; then

$$\frac{1}{h}\int_0^h |g(\eta+t)|dt = \frac{1}{h}\int_0^h |f(\xi-t)|dt = o(h^{-c})$$

as $h \to 0+$. Also $I^c g(\eta)$ exists since $J^c f(\xi)$ does, by (6); so by Theorem 2 η is a right L-point of $I^c g$. Since

$$\frac{1}{h}\int_0^h |J^c f(\xi-s) - J^c f(\xi)|ds = \frac{1}{h}\int_0^h |I^c g(\eta+s) - I^c g(\eta)|ds \to 0$$

as $h \to 0+$, ξ is a left L-point of $J^c f$, as required.

6. Two-sided fractional integrals.

The lemmas of §5 enable us to make the following synthesis of Theorems 1 and 2, involving two-sided Lebesgue points.

Theorem 3. If $0 < c < 1$ and $f \in L(a,b)$, then the L-points of

$$K^c f(x) = \int_a^b \frac{|x-t|^{c-1}}{\Gamma(c)} f(t)dt$$

in (a,b) are just the points x at which $K^c f(x)$ exists.

Proof. Every L-point is a point of existence, by the definition (3). For the converse, suppose that $\xi \in (a,b)$ and that $K^c f(\xi)$ exists. Then $I^c f(\xi)$ and $J^c f(\xi)$ exist, and their sum is $K^c f(\xi)$.

By Lemma 1,

$$\frac{1}{h}\int_0^h |f(\xi-t)|dt = o(h^{-c}) \qquad \text{as} \qquad h \to 0+,$$

so by Lemma 4 ξ is a left L-point of J^cf. And by Lemma 3 ξ is a right L-point of J^cf. Thus ξ is a L-point of J^cf.

By Lemma 2,

$$\frac{1}{h}\int_0^h |f(\xi+t)|dt = o(h^{-c}) \qquad \text{as} \qquad h \to 0+,$$

so by Theorem 2 ξ is a right L-point of I^cf. And by Theorem 1 ξ is a left L-point of I^cf. Thus ξ is a L-point of I^cf.

Since $I^cf(x) + J^cf(x) = K^cf(x)$ for almost all $x \in (a,b)$,

$$\frac{1}{h}\int_0^h |K^cf(\xi \pm s) - K^cf(\xi)|ds$$

$$\leq \frac{1}{h}\int_0^h |I^cf(\xi \pm s) - I^cf(\xi)|ds + \frac{1}{h}\int_0^h |J^cf(\xi \pm s) - J^cf(\xi)|ds;$$

these all tend to zero as $h \to 0+$, and so ξ is a L-point of K^cf, as required.

7. Reference.

[1] G.H. Hardy and J.E. Littlewood, Some properties of fractional integrals. I. Math. Zeitschr. 27*(1928), 565-606.

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