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SOME REMARKS ON DIFFERENTIAL EQUIVALENCE

The main purpose of this article is to answer some questions arising from the articles of S. Leader [6], [7]. Using the Henstock-Kurzweil integral he introduces a definition of the "differential" df of a function f. This concept is rather broad with not too many properties in general. In order to develop a nice theory a class of functions, called "dampable", is introduced and it is shown that most of the familiar calculus manipulations with differentials can be verified for this class of functions in a satisfying and natural manner. Neither article characterizes this class, and it is our purpose here to give that characterization.

The characterization will be no surprise. Just as for the Lebesgue integral the classes of VB and AC functions arise with compelling regularity, in any study of the Henstock-Kurzweil (alias Denjoy-Perron) integral the classes of VBG_* and ACG_* functions intrude everywhere. Indeed Ward [14] in his study of the Perron-Stieltjes integral, which is intimately related to these matters, suggests that the class of VBG_* functions is the largest class that should arise in these kind of matters.

The key concept needed in presenting this material is the notion of "differential equivalence" (in the language of Kolmogorov [5]) or "variational equivalence" (in the language of Henstock [4]). This is a true equivalence relation and the equivalence classes are what Leader calls his "differentials". In the first section we sketch the apparatus needed for this presentation, in what appears to be a convenient and useful language. Most of the terminology is modelled after standard sources; for example the term "covering relation" is taken from Federer [2]. Proofs here are omitted but may be constructed from the material in [12] and [13]. Section two contains a brief account of the notion of differential equivalence and some basic differentiation results; the proofs of the main results (2.8), (2.9), and (2.10) are given in detail. Finally section three then contains the characterization of dampable functions and its proof.

§1. Notation and preliminaries. Throughout $[a, b] \subset R$ is a fixed interval and all functions are real-valued functions defined on that interval.

(1.1) A <u>covering relation</u> is a collection of pairs (I, x) where I is a closed subinterval of $[a, b]$ and $x \in I$.

For convenience the collection of all such closed intervals may be called $\mathcal{I}$ and so an covering relation is a subset of the product $\mathcal{I} \times [a, b]$.

(1.2) If β is an covering relation and E is a set of real numbers then $\beta(E)$ and $\beta[E]$ denote the following sets:

(i) $\beta(E) = \{ (I, x) \in \beta : I \subset E \}$,

(ii) $\beta[E] = \{ (I, x) \in \beta : x \in E \}$.

The expressions $\beta(E)$ and $\beta[E]$ are also covering relations, and in fact subsets of β. The passage to $\beta(E)$ and $\beta[E]$ from β is a common device in the theory and this notation is a convenient one. In some settings $\beta(E)$ is called a "pruning" of β where the language is meant to indicate that some inessential members of β have been removed.

(1.3) A <u>packing</u> is a finite covering relation π with the property that for distinct pairs (I_1, x_1) and (I_2, x_2) belonging to π the intervals I_1 and I_2 do not overlap.

The most important packings are those that form partitions. Conventionally a partition of an interval is a finite collection of non-overlapping subintervals that covers the interval; here we use the same word to denote that idea but with the associated points incorporated into the concept.

(1.4) A packing π is said to be a <u>partition</u> of the interval $[a, b]$ provided

$$\bigcup_{(I, x) \in \pi} I = [a, b] .$$

(1.5) A covering relation β is said to be a <u>full covering relation</u> at a point x provided that there exists a $\delta > 0$ so that

$$([x,y], x) \in \beta \qquad \text{for every} \qquad y \in (x, x+\delta)$$

and

$$([y, x], x) \in \beta \qquad \text{for every} \qquad y \in (x-\delta, x).$$

Such a relation is said to be a full covering relation on a set E if it is a full covering relation at each point of E.

These covering relations have been chosen so as to correspond to the theory of ordinary limits and ordinary derivatives. For the study of approximate derivatives or symmetric derivatives there would be an appropriate version of the covering relations needed. We give also a relation that is dual to this; the duality is expressed in (1.7).

(1.6) A covering relation β is said to be a <u>fine covering relation</u> at a point x provided that for every $\varepsilon > 0$ either there is a point y with $x < y < x + \varepsilon$ and $([x,y], x) \in \beta$, or else there is a point y with $x > y > x - \varepsilon$ and $([y, x], x) \in \beta$. Again β is a fine covering relation on a set E if it is a fine covering relation at each point of E.

The duality between full and fine covering relations is expressed by the following result.

(1.7) Let β be a covering relation and let

$$\alpha = \{ (I, x) : I \subset [a, b], x \in I, (I, x) \text{ not in } \beta \}.$$

Then α is a full covering relation at a point x if and only if β is not a fine covering relation at x.

A covering theorem is a statement about subsets of covering relations, usually to the effect that some subset has a specified property, or that a subset exists with a specified property. We state some simple covering theorems needed. Deeper results will depend on essentially deeper covering theorems such as the Vitali covering theorem, for example.

(1.8) Let β_1 and β_2 be full covering relations on a set E. Then $\beta_1 \cap \beta_2$ is a full covering relation on E.

(1.9) Let β_1 be a full covering relation on a set E and let β_2 be a fine covering relation on E. Then $\beta_1 \cap \beta_2$ is a fine covering relation on E.

(1.10) Let β be a full [fine] covering relation on a set E and let G be an open set containing E. Then $\beta(G)$ is a full [fine] covering relation on E.

(1.11) Let β_α be a full [fine] covering relation on a set E_α for each $\alpha \in A$. Then

$$\beta = \bigcup_{\alpha \in A} \beta_\alpha$$

is a full [fine] covering relation on the union of the $\{ E_\alpha \}$.

The covering theorems that are available for full and fine covering relations are among our most used tools in the general theory that is developed. The first of these, due apparently to Pierre Cousin [1] in 1895, is equivalent to the Bolzano-Weierstrass theorem; it seems to be doomed to a history of frequent rediscovery (cf. [3], [8], [9] and [11]). The second is a useful way of expressing as a covering theorem a common device in analysis.

(1.12) [P. Cousin] Let β be a full covering relation on an interval [c, d]. Then β contains a partition of the interval [c, d].

(1.13) Let β be a full covering relation on a set E. Then there is a disjointed sequence of sets $\{E_1, E_2, E_3, \ldots\}$ with union E that has the following property: if $x \in E_n$ and

$$\inf E_n < z < x < y < \sup E_n$$

then both pairs $([x, y], x)$ and $([z, x], x)$ belong to β.

We turn now to a discussion of the functions that arise in this study. An <u>interval function</u> is a real-valued function whose domain is the collection of all closed subintervals of our fixed interval [a, b]. Functions defined on [a, b] shall be called <u>point functions</u> in order to distinguish the notions.

(1.14) If f is a point function then the interval functions Δf and $|\Delta f|$, associated with f are defined as follows:

$$\Delta f([c, d]) = f(d) - f(c) ,$$

and

$$|\Delta f|([c, d]) = |f(d) - f(c)| ,$$

(1.15) For the identity function on $[a, b]$, $f(x) \equiv x$, we shall use the notation Δx to denote the increment Δf. Thus $\Delta x(I) = |I|$ is just the length of the interval I.

The notion of an interval function, as important as it is for many of our applications, does not offer quite the flexibility that we require. Instead we shall need frequently to use functions whose domains are covering relations.

(1.16) An <u>interval-point function</u> is a real-valued function whose domain is an covering relation.

Generally we may assume that an interval-point function is defined on the entire product

$$\mathcal{I} \times [a, b] = \{ (I, x) : I \text{ an interval}, x \in [a, b] \} ,$$

although, since we shall work here exclusively with full and fine covering relations, such an h need be defined only for pairs $([x, y], x)$ and $([x, y], y)$ where the associated point is at an endpoint. If h is an interval function then it shall be considered as well as an interval-point function by agreeing that $h(I, x) = h(I)$. If h is an interval-point function and f is a point function then the product fh shall be considered as the interval-point function

$$fh : (I, x) \to (fh)(I, x) = f(x)h(I, x).$$

We require a limit concept for interval-point functions.

(1.17) Let h be an interval-point function. Then at a point x_0 we write

$$\lim_{I \to x_0} h(I, x) = c$$

provided

$$\lim_{t \to 0+} h([x_0, x_0 + t], x_0) = \lim_{t \to 0+} h([x_0 - t, x_0], x_0) = c .$$

Such a number c (including the case $c = \pm\infty$) is called the <u>limit</u> of h at x_0. The extreme limits $\limsup_{I \to x_0}$ and $\liminf_{I \to x_0}$ are similarly defined. A number c (including $\pm\infty$) for which there are positive numbers $t_1, t_2, \ldots$ such that $t_n \to 0$ for which

$$h([x_0, x_0 + t_n], x_0) \to c$$

or

$$h([x_0 - t_n, x_0], x_0) \to c .$$

is called a <u>weak limit</u> of h at x_0. Continuity and weak continuity of interval-point functions at a point x_0 are defined in the obvious manner using these limits.

The study of interval-point functions focusses mainly on the limit properties of such functions taken together with their variation properties. We define firstly a variation taken relative to covering relations, and then relative to the families of full and fine covering relations.

(1.18) Let β be an covering relation and let h be an interval-point function. Then by $\mathrm{Var}(h, \beta)$ we denote

$$\mathrm{Var}(h, \beta) = \sup \left\{ \sum_{(I,x) \in \pi} | h(I, x) | : \pi \subset \beta, \ \pi \text{ a packing} \right\}$$

We refer to this as the <u>variation</u> of the function h relative to the covering relation β.

(1.19) Let h be an interval-point function and E a set of real numbers. Then by $V^*(h, E)$ and $V_*(h, E)$ we denote

$$V^*(h, E) = \inf \{ \mathrm{Var}(h, \beta) : \beta \text{ a full covering relation on } E \},$$
$$V_*(h, E) = \inf \{ \mathrm{Var}(h, \beta) : \beta \text{ a fine covering relation on } E \} .$$

It is convenient to have some additional suggestive notation for these two variations. We write h^* and h_* for the two set functions

$$h^*(E) = V^*(h, E) \quad \text{and} \quad h_*(E) = V_*(h, E) .$$

If some nomenclature is required we might refer to these as the full and fine variational outer measures generated by h. It should be remarked at this stage that these outer measures depend directly on the covering relations used. One might wish to study other types of covering relations in which case other outer measures would arise. Our study focusses on these two outer measures because they express properties related to the differentiation and integration of functions, with the derivative understood in the "ordinary" sense.

The elementary properties of the variation are developed in the ensuing statements.

(1.20) Let h be an interval-point function and $x_0 \in [a, b]$. Then $h^*(\{x_0\})$ is

$$\limsup_{t \to 0+} h([x_0, x_0+t], x_0) + \limsup_{t \to 0+} h([x_0-t, x_0], x_0)$$

and $h_*(\{x_0\})$ is

$$\min \{ \liminf_{t \to 0+} h([x_0, x_0+t], x_0), \liminf_{t \to 0+} h([x_0-t, x_0], x_0) \}$$

(1.21) An interval-point function h is continuous at a point x_0 if and only if $h^*(\{x_0\}) = 0$ and is weakly continuous at a point x_0 if and only if $h_*(\{x_0\}) = 0$.

(1.22) $(h_1 + h_2)^* \leq h_1^* + h_2^*$.

This is an easy consequence of the covering theorem (1.8). Since fine covering relations do not have this property the same relation cannot be obtained for the fine outer measures. For example take $h_1(I)$ to be $|I|$ if this is rational and 0 otherwise, and take $h_2(I) = |I| - h_1(I)$. Then $(h_1+h_2)_*$ will give a nonvanishing outer measure (in fact Lebesgue outer measure) while both $(h_1)_*$ and $(h_2)_*$ vanish.

(1.23) $h_* \leq h^*$.

(1.24) Let h be a subadditive interval function of bounded variation on the interval $[a, b]$ and let Vh be its total variation function. Then Vh is an additive interval function, and

$$V^*(Vh - h, [a, b]) = 0.$$

The full and fine variations h^* and h_* asssociated with an interval-point function h are genuine outer measures on the real line that have nice topological properties.

(1.25) For any interval-point function h the set functions h^* and h_* are metric outer measures on $[a, b]$.

The regularity behaviour has not been commented upon in the literature. Let us mention, without proofs, the following facts.

(1.26) In general h^* and h_* need not be Borel regular even if h is an interval function.

(1.27) Suppose that h is a continuous interval function. Then the outer measure h^* is $F_{\sigma\delta}$ regular.

If g is a continuous, nondecreasing function on the interval $[a, b]$ then the outer measure Δg^* corresponds to the usual Lebesgue-Stieltjes outer measure on $[a, b]$ generated by g. This in turn could lead to a construction of the Lebesgue-Stieltjes integral by the standard devices of measure theory. In the present setting this will correspond to natural variational computations.

(1.28) Let g be continuous and nondecreasing, let f be a nonnegative Δg^*-measurable point function f, and write

$$F(x) = \int_a^x f(t)\, dg(t)$$

as the Lebesgue-Stieltjes indefinite integral of f. Then

$$V^*(\Delta F - f\Delta g, [a, b]) = 0$$

and

$$\Delta F^*(E) = (f\Delta g)^*(E) = \int_a^b f(t)\chi_E(t)\, dg(t)$$

for any Δg^*-measurable set E.

§2. Differential equivalence. Most of the results that are developed in this theory concern derivatives and integrals of interval-point functions, and are identical for those interval-point functions that belong to the same differential equivalence class. The terminology for this equivalence relation and the general idea of exploiting it, in a related setting, are due to Kolmogorov. Henstock uses the same idea in his concept of "variational" equivalence.

(2.1) Let h_1 and h_2 be interval-point functions. We say that h_1 and h_2 are <u>differentially equivalent</u> and we write $h_1 \equiv h_2$ provided that

$$V^*(h_1 - h_2, [a, b]) = 0 .$$

The fundamental properties are listed below.

(2.2) Let h_1 and h_2 be interval-point functions. If $h_1 \equiv h_2$ then, for any point function f, $fh_1 \equiv fh_2$.

(2.3) Let h_1, h_2, and k be interval-point functions. If $h_1 \equiv h_2$ and

$$\limsup_{I \to x} |k(I, x)| < +\infty$$

everywhere, then $kh_1 \equiv kh_2$.

(2.4) Let h be an interval-point function and f a point function. Then $fh \equiv 0$ if and only if $f(x) = 0$ for h^*-almost every point x.

(2.5) Let h be an interval-point function and f,g point functions. Then $fh \equiv gh$ if and only if $f(x) = g(x)$ for h^*-almost every point x.

(2.6) Let f be a point function. Then $\Delta f \equiv 0$ if and only if f is constant.

(2.7) Let h_1 and h_2 be interval-point functions. If $h_1 \equiv h_2$ then $(h_1)^* = (h_2)^*$ and $(h_1)_* = (h_2)_*$.

The remainder of this section is devoted to some concerns regarding the differentiation of interval-point functions. The relation between differentiation and integration on the real line is commonly refered to as the fundamental theorem of the calculus. In the presentation here another viewpoint may be taken: rather than first developing an integration theory we can use the central concept of differential equivalence to obtain our relation. Then the version for the integration theory will follow as a corollary. Roughly the theorem asserts that the differential equivalence

$$h \equiv fk$$

for a pair of interval-point functions h and k, and a point function f is equivalent to the fact of f being the derivative of h with respect to k.

The notation for the derivative of an interval-point h with respect to another interval-point k might be taken as

$$D_k\, h(x) = \lim_{I \to x} \frac{h(I, x)}{k(I, x)}$$

with the understanding that a zero denominator forbids the existence. Since all differentiation statements in this section are made k_*-almost everywhere this vanishing denominator plays no role. We shall avoid the derivation notation and express our results as limit theorems, with the understanding that the motivation is in terms of derivatives. These results are rather compactly and generally expressed and so the proofs shall be given.

(2.8) Let h and k be interval-point functions, f a point function and suppose that $h \equiv fk$. Then at k_*-almost every point x

$$\lim_{I \to x} \frac{h(I, x)}{k(I, x)} = f(x)$$

PROOF. For each integer n let

$$\beta_n = \left\{ (I, x) : \quad |h(I, x) - f(x)k(I, x)| \geq \frac{|k(I, x)|}{n} \right\}$$

and let Y_n denote the set of all points y at which the collection β_n is a fine covering relation. Let Y denote the union of the sequence of sets Y_n; we shall show that $k_*(Y) = 0$ and that for every point x not in Y the limits stated in the theorem must hold.

Let $\varepsilon > 0$ be given. Since $h \equiv fk$ we may select a full covering relation α on $[a, b]$ so that

$$\mathrm{Var}(h - fk, \alpha) < \varepsilon .$$

By the covering theorem (1.9) each $\alpha \cap \beta_n$ is a fine covering relation on Y_n and hence

$$\begin{aligned} k_*(Y_n) &= V_*(k, Y_n) \leq \mathrm{Var}(k, \alpha \cap \beta_n) \\ &\leq \mathrm{Var}(n(h - fk), \alpha \cap \beta_n) \leq n\, \mathrm{Var}(h - fk, \alpha) \leq n\varepsilon . \end{aligned}$$

Since ε is arbitrary it follows that each Y_n has k_*-measure zero and so $k_*(Y) = 0$ as stated.

Now for each integer n define the collection of interval-point pairs α_n by

$$\alpha_n = \left\{ (I, x) : \quad |h(I, x) - f(x)k(I, x)| < \frac{|k(I, x)|}{n} \right\}.$$

By the way in which the sets Y_n were defined and because of (1.7) each α_n must be a full covering relation on the set $[a, b] \setminus Y$. Thus at each point x in $[a, b] \setminus Y$ we easily verify the required limits.

(2.9) Let h and k be interval-point functions, let f be a point function and suppose that the outer measure k^* is σ-finite. Suppose that the limit

$$\lim_{I \to x} \frac{h(I, x)}{k(I, x)} = f(x)$$

holds both h^*-almost everywhere and k^*-almost everywhere. Then $h \equiv fk$.

PROOF. Let X be the set of points x at which the above stated equality holds. Then for every integer n the collection

$$\beta = \{ (I, x) : |h(I, x) - f(x)k(I, x)| \leq n^{-1} |k(I,x)| \}$$

must be a full covering relation on X. From this one deduces that

$$V^*(h - fk, Y) \leq n^{-1} k^*(Y)$$

for every subset Y of X. As n is arbitrary and K^* is σ-finite on X we must have

$$V^*(h - fk, X) = 0 .$$

Using (1.25), (2.2) and the fact that h^* and k^* vanish on the complement of X, we obtain

$$V^*(h - fk, [a, b]) \leq V^*(h - fk, X) + h^*([a, b] \setminus X)$$
$$+ (fk)^*([a, b] \setminus X) = 0$$

which gives, by definition, the required assertion $h \equiv fk$.

(2.10) Let h and k be interval-point functions, suppose that k^* is σ-finite on a set E, and that for any real numbers c_1 and c_2 we have

$$(c_1h + c_2k)^* = (c_1h + c_2k)_*$$

on the set E. Then at h^*- and k^*-almost every point x in E either the limit

$$\lim_{I \to x} h(I, x)/(k(I, x)$$

exists finitely or else

$$\lim_{I \to x} | h(I, x)/k(I, x) | = +\infty .$$

If h^* is also σ-finite on E then at k^*-almost every point the limit is finite.

PROOF. (In the limit given here we may interpret a quotient $c/0$ as 0 if $c = 0$, as $+\infty$ if $c > 0$ and as $-\infty$ if $c < 0$.) Without loss of generality we may suppose that $k^*(E) < +\infty$. Let X denote the set of points x in E at which the stated limits do not exist. We observe that at each point x in X the ratio $h(I, x)/k(I, x)$ must have two different limit numbers, one of which is finite. For rational numbers p, r, and s with $r < s < p$ let X_{prs} denote the set of points x at which there is a limit number less than r and a limit number in the interval (s, p). We will show that

$$h^*(X_{prs}) = k^*(X_{prs}) = 0 . \tag{2.10.1}$$

Let β be any full covering relation on X_{prs} and define the collections

$$\beta_1 = \left\{ (I, x) : \frac{h(I, x)}{k(I, x)} < r \right\},$$

$$\beta_2 = \left\{ (I, x) : s < \frac{h(I, x)}{k(I, x)} < p \right\}.$$

By our assumptions on the limit numbers of this quotient we see that β_1 and β_2 are both fine covering relations on X_{prs}. We shall use these to establish that

$$(h - pk)^*(X_{prs}) = k^*(X_{prs}) = 0 \tag{2.10.2}$$

and from this we may then deduce (2.10.1). If $(I, x) \in \beta_1$ then

$$|h(I, x) - pk(I, x)| \geq (p - r)\ |k(I, x)|$$

so that

$$(p - r)k_*(X_{prs}) \leq Var(h - pk, \beta \cap \beta_1) \leq Var(h - pk, \beta) .$$

Similarly if $(I, x) \in \beta_2$ then

$$|h(I, x) - pk(I, x)| \leq (p - s)\ |k(I, x)|$$

so that

$$(h - pk)_*(X_{prs}) \leq Var(h - pk, \beta \cap \beta_1) \leq (p - s)Var(k, \beta) .$$

From these we deduce that

$$(h - pk)^*(X_{prs}) \geq (p - r)\ k_*(X_{prs}) \tag{2.10.3}$$

and

$$(h - pk)_*(X_{prs}) \leq (p - s)\ k^*(X_{prs}) . \tag{2.10.4}$$

Since $(h - pk)^* = (h - pk)_*$ and $k^* = k_*$ on the set E and the numbers in (2.10.3) and (2.10.4) are finite, this can occur only if (2.10.2) is valid, and then (2.10.1) must follow.

The set X may be expressed as a denumerable union of sets of such a type and so must have zero h^*- and k^*-measure. To complete the proof we have only to show that, under the additional hypothesis that h^* is σ-finite on E, the infinite limits can occur only on a set of k^*-measure zero. Indeed we can establish the following fact under weaker hypotheses than in (2.10). The proof is then complete.

(2.10.3) If h^* is σ-finite on E then
$$\limsup_{I \to x} |h(I, x)/k(I, x)| < +\infty$$
at k_*-almost every point x in E.

We may suppose that $h^*(E) < +\infty$. Then the set
$$E_0 = \{ x \in E : \limsup_{I \to x} |h(I, x))/(k(I, x)| = +\infty \}$$
will have $k_*(E_0) = 0$: to see this let $N > 0$, let β be a full covering relation on E and define the collection
$$\beta_1 = \{ (I, x) : |h(I, x)/k(I, x)| > N \} .$$
Certainly β_1 is a fine covering relation on E_0 so that
$$k_*(E_0) \leq \mathrm{Var}(k, \beta_1 \cap \beta) \leq N^{-1} \mathrm{Var}(h, \beta) .$$
This gives $k_*(E_0) \leq N^{-1} h^*(E_0)$ from which $k_*(E_0) = 0$ evidently follows.

§3. Dampable functions. We pass now to a study of the properties of additive interval functions Δf with regards to this notion of differential equivalence. The first two results are known and may be found in [12, pp. 194-195] and [13, p.97]. The first is an application of the Vitali covering theorem and the second of the differentiation theorem (2.10).

(3.1) Let f be continuous and VBG_* on the interval $[a, b]$. Then the full and fine variational outer measures Δf_* and Δf^* are identical.

(3.2) Let f and g be continuous, VBG_* functions on the interval $[a, b]$. Then the derivative

$$\lim_{I \to x} \frac{\Delta f(I)}{\Delta g(I)}$$

exists finitely or infinitely Δf^*- and Δg^*-almost everywhere in $[a, b]$.

Let us now give the definitions of Leader [7]. They here assume a slightly different form and we adopt a modified terminology.

(3.3) Let h be an interval-point function. We say that h is (VB)-dampable if there is a positive point function k and a function of bounded variation g so that $k\,h \equiv \Delta g$. We say that h is (AC)-dampable if there is a positive point function k and an absolutely continuous function g so that $k\,h \equiv \Delta g$.

The main theorems of this section provide characterizations of these notions.

(3.4) Let f be a continuous point function on the interval $[a, b]$. Then the following are equivalent.

(i) Δf is (VB)-dampable.

(ii) Δf^* is σ-finite on $[a, b]$.

(iii) f is VBG_* on $[a, b]$.

(3.5) Let f be a continuous point function on the interval $[a, b]$. Then the following are equivalent.

(i) Δf is (AC)-dampable.

(ii) Δf^* is σ-finite on $[a, b]$ and $\Delta f^*(N) = 0$ for every set N of Lebesgue measure zero.

(iii) f is ACG_* on $[a, b]$.

PROOF. Let us prove (3.4). The equivalence of (ii) and (iii) is well known (see [12, p.186] and [13, p.94]), and it is easy to show that (i) implies (ii). Thus the real content of the proof rests in taking a continuous, VBG_* function f and constructing a suitable positive point function k and a bounded variation function g so that $k\Delta f \equiv \Delta g$.

By a theorem of Ward (see Saks [10, pp.236-237]) there must exist a continuous, strictly increasing function g_1 so that the extreme derivates

$$\limsup_{I \to x} \; |\Delta f(I)/\Delta g_1(I)| \quad < +\infty$$

at each point x. By using in addition (3.2) we see that in fact the limit

$$\lim_{I \to x} \frac{\Delta f(I)}{\Delta g_1(I)} = k_1(x)$$

exists finitely at each x except in a set N_1 for which $\Delta g_1^*(N_1) = \Delta f^*(N_1) = 0$. By (2.9), this gives the relation

$$V^*(k_1 \Delta g_1 - \Delta f, [a, b] \setminus N_1) = 0 .$$

Let now N_2 denote the set of points x at which $k_1(x)$ exists and is zero. Certainly $\Delta f^*(N_2) = 0$. Define the point function k by defining $k(x)$ as 1 if x is in N_1 or N_2, as $1/k_1(x)$ if this is positive and as $-1/k_1(x)$ if this is positive with $x \in [a, b] \setminus (N_1 \cup N_2)$. Write M_1 for the set of points x not in N_1 at which $k_1(x)$ is positive and M_2 for the set of points x not in N_1 at which $k_1(x)$ is negative. Let χ_{M_1} and χ_{M_2} denote the indicator functions for M_1 and M_2. This gives

$$V^*(k\Delta f - \chi_{M_1} \Delta g_1 + \chi_{M_2} \Delta g_1, [a, b])$$

$$\leq V^*(k(\Delta f - k_1 \Delta g_1), M_1) + V^*(k(\Delta f - k_1 \Delta g_1), M_2) + V^*(k\Delta f, N_1) + V^*(k\Delta f, N_2)$$
$$= 0.$$

So we have that $k\Delta f \equiv \chi_{M_1} \Delta g_1 - \chi_{M_2} \Delta g_1$, and it remains only to show that there is a function g of bounded variation so that Δg is in turn equivalent to these.

The function g is taken by using the Lebesgue-Stieltjes integrals

$$g(x) = \int_a^x \chi_{M_1}(t)\, dg_1(t) - \int_a^x \chi_{M_2}(t)\, dg_1(t)$$

as in (1.28). Note that this requires proving that M_1 and M_2 are Δg_1^*-measurable, but it is a straightforward matter to check that both are in fact Borel sets.

The proof of (3.5) is similarly obtained. One needs as well the fact that Δx^* is Lebesgue outer measure on [a, b] and that Wards theorem for ACG_* functions permits g_1 to be taken as absolutely continuous.

As a corollary to these two theorems we have the following which provides an answer to a question posed by Leader [6, p.353].

(3.6) Let f be a continuous function on the interval [a, b]. Then Δf is (VB)-dampable [(AC)-dampable] if and only if $|\Delta f|$ is.

PROOF. Suppose that Δf is (VB)-dampable. Then there are k and g as in (3.3) so that $k\Delta f \equiv \Delta g$. Let v be the total variation function for g. Then $|\Delta g| \equiv \Delta v$ by (1.24) and $|k\Delta f| \equiv |\Delta g|$ by the triangle inequality, so that $k|\Delta f| \equiv \Delta v$ and we have, as required, that $|\Delta f|$ is (VB)-dampable.

Conversely if $|\Delta f|$ is (VB)-dampable then $k|\Delta f| \equiv \Delta g$ for a positive point function k and a function g of bounded variation. By (2.2) and (27) this gives $\Delta f^* = (k^{-1}\Delta g)^*$; the outer measure Δf^* must accordingly be finite on each set

$$X_n = \{ x \in [a, b] : k(x) > 1/n \} ,$$

which exhibits Δf^* as σ-finite on [a, b]. By theorem (3.4) we have that Δf is (VB)-dampable.

For the (AC)-dampable case the arguments are similar.

Finally let us conclude with a query. These same concerns arise in altered settings. For example, just on the real line, one can replace the notions of full and fine covering relations with the analogous ones which handle the approximate derivative or the symmetric derivative. What are the characterizations of (VB)-dampable in these settings? More importantly, perhaps, these questions are of some importance in higher dimensions where there are a host of important covering relations that should be studied. For any of these what is the characterization of the dampable functions?

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