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# Borel measurability of extreme path derivatives

The derivative F' of any differentiable function F is a function of Baire class one, since F' is the pointwise limit of a sequence of continuous functions $\{n(F(n+\frac{1}{n})-F(x))\}_{n=1}^{\infty}$. A problem that dates back to the beginning of this century is that of finding the Baire classification of various types of extreme derivatives. Sierpinski [11] showed that the Diní derivatives of a function of Baire class $\alpha$ are in Baire class $\alpha + 3$. Banach [2] proved that the Dini derivatives of the bounded functions of Baire class $\alpha$ are in Baire class $\alpha + 2$. We have also by Kempisty [5] and Hajek [4] the successive results that the extreme bilateral derivatives (for arbitrary functions) are in Baire class 3 and in Baire class two. Misik [7] was able to generalize Banach's result for arbitrary functions of Baire class $\alpha$. He showed that the upper (lower) Dini derivatives of a Borel function of Baire class $\alpha$ are upper (lower) semi-Borel functions of Baire class $\alpha + 1$. He also [8], [9] proved that for any ordinal number $\alpha$ the upper (lower) unilateral approximate derivatives of Borel functions of the class $\alpha$ are lower (upper) semi-Borel functions of the class $\alpha + 2$.

Bruckner, O'Malley and Thomson [3] introduced the concept of path derivative as a unifying approach to the study of a number of generalized derivatives. They showed that for a system of paths

$E = \{E_x : x \in R\}$ satisfying the external intersection condition any E-derivative is in Baire class one. We begin this paper with the definition of a continuous system of paths, then we show that the extreme path derivative of a continuous function relative to such systems of paths is a function in Baire class two. Also we show that under some added condition the extreme path derivatives of a function in Baire class one is in Baire class four. This result does not hold in general for an arbitrary Borel measurable function. In fact we give an example of a continuous system of paths E and a function F in Baire class two such that $\overline{F}'_E$ is not Borel measurable. It will also be shown that the extreme path derivatives of a Borel measurable function with respect to a continuous system of paths is Lebesgue measurable. We conclude the paper by briefly discussing the Borel measurability of path derivatives and proving that the path derivative of a function of Baire class $\alpha$ is in Baire class $\alpha + 2$ when the system of path is continuous.

The continuous functions are said to be of type $B_0$ (Baire class 0) Functions which are limits of convergent sequences of continuous functions are of type $B_1$ (Baire class 1). Let $\Omega$ be the first non-denumerable ordinal number. For any $\alpha < \Omega$ if the functions of types $B_\beta$ have been defined for every $\beta < \alpha$, then the functions of type $B_\alpha$ (Baire class $\alpha$) are limits of convergent sequences of functions of types $\beta < \alpha$. By transfinite induction this defines the Baire functions of all classes $\alpha < \Omega$.

Let A and B be compact subsets of R (the real line). Then

$$d_H(A,B) = \inf\{\varepsilon: A(\varepsilon)\supset B \text{ and } B(\varepsilon)\supset A\},$$

where

$$A(\varepsilon) = \bigcup_{x\in A} (x-\varepsilon,x+\varepsilon).$$

In the set of compact subsets of R, $d_H$ is a metric called Hausdorff metric.

If $\Phi$ is a function from a given topological space X into the space of all non-void subsets of a given topological space $y$, then a selection for $\Phi$ is any function F from X into $y$ such that $F(x)\in\Phi(x)$ for all $x\in X$. A selection is called Borel measurable or measurable depending on whether F is Borel measurable or measurable.

A set $E_x$ is a path at x for $x\in R$ if $E_x \subseteq R$, $x\in E_x$, and x is a limit point of $E_x$. A system of paths E, is a collection $\{E_x: x\in R\}$, where each $E_x$ is a path at x. If $\lim\limits_{\substack{y\to x\\ y\in E_x}} \frac{F(y)-F(x)}{y-x}$ exists and is finite, then it is called the E-derivative of F at x and is denoted by $F_E'(x)$. The extreme E-derivatives are similarly defined. When we are dealing with a specific system of paths tailored to a continuous function, we may alter $E = \{E_x: x\in[0,1]\}$ to $E_1 = \{\overline{E}_x: x\in[0,1]\}$, where $\overline{E}_x$ denotes the closure of $E_x$. Throughout this paper $\tilde{A}$ denotes the complement of A, N the set of positive integers, and Q the set of rational numbers.

Definition 1: Let $E = \{E_x:x \in [0,1]\}$ be a system of paths, with each $E_x$ being compact. If the function $E: x \to E_x$ is a continuous function, we say E is a continuous system of paths. (E with the Hausdorff metric forms a metric space.)

Definition 2: Let $E = \{E_x: x \in [0,1]\}$ be a system of paths:

(i) E is said to be bilateral at x if x is a bilateral point of accumulation of $E_x$.

(ii) E is said to be unilateral at x if x is a unilateral point of accumulation of $E_x$.

Later in the paper we will obtain a generalization of the following theorem due to Sierpinski.

Theorem 3 (Sierpinski): If F is continuous on [a,b], then each of the Dini derivative is in Baire class 2.

Proof: We prove the theorem for $D^+F$, a similar proof holds for $D^-F$, $D_+F$, and $D_-F$.

For each positive integer n let

$$F_n(x) = \sup_{t\in[a,b]} \left\{\frac{F(t)-F(x)}{t-x} : x + \frac{1}{n+1} \leq t \leq x + \frac{1}{n}\right\}.$$

Since F is continuous, each function $F_n$ is also continuous. It is easy to verify that $D^+F(x) = \limsup_{n\to\infty} F_n(x)$. But an upper limit of a sequence of continuous functions is in Baire class 2.

Example 4 ([3], Theorem 3.1 page 100): There is a continuous function F such that, given any function f on R, a system of paths E = $\{E_x: x \in R\}$ can be found so that $F'_E = f$.

Example 4 implies that the extreme path derivatives of a continuous function could behave badly. So in order to have a nice extreme path derivative we should have some restrictions on the system of paths as well as the function.

Main theorem 5: let $E = \{E_x: x \in [0,1]\}$ be a continuous system of paths, $F(x)$ a continuous function defined on $[0,1]$.

(a) If $F$ is $E$-differentiable, then $F_E'(x)$ is a function in Baire class one.

(b) $\overline{F}_E'$ and $\underline{F}_E'$ are functions of Baire class two.

In order to prove theorem 5, one would like to imitate Sierpinski's proof. We have two immediate problems. The first problem is that if we define

$$F_n(x) = \sup \{\frac{F(y)-F(x)}{y-x} : y \in E_x \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]\},$$

then $E_x \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]$ might be empty. In that case what should we define for $F_n(x)$?.

The second problem is that $E_x$ and $E_y$ might behave differently when $x$ and $y$ are very close. In fact Example 4 illustrates that even for a continuous function $F$ it is possible to find a system of paths such that $\overline{F}_E'$ is not measurable. Thus, in order to achieve our aim, we will attempt to choose a sequence of positive real numbers $\{a_n\}_{n=1}^{\infty}$, $a_{n+1} < a_n$ and $\lim_{n\to\infty} a_n = 0$ such that $E_x \cap [x + a_{n+1}, x + a_n] \neq \emptyset$ for all $x$. The following lemma shows that this is possible when the system of paths $E = \{E_x: x \in [0,1]\}$ is continuous. The foundations of the proof of Theorem 5 are based on Lemmas 8 and 9.

Lemma 6: If $E = \{E_x: x \in [0,1]\}$ is a continuous system of paths. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n > 0$ for all $n$, $\{a_n\}$ decreases to zero, and

$(E_x \cap [x+a_{n+1}, x+a_{n-1}]) \cup (E_x \cap [x-a_{n-1}, x-a_{n+1}]) \neq \emptyset$ for all $x \in [0,1]$.

Proof: Let $a_1 = 1$.

Define

$$a_2 = \tfrac{1}{2} \inf_{x \in [0,1]} \sup\{|y| : y \in ((E_x - x) \cap [0,a_1)) \cup ((E_x - x) \cap [-a_1,0])\}.$$

Obviously $0 \leq a_2 < a_1$. Let $a_n$ be defined. Inductively define

$$a_{n+1} = \tfrac{1}{2} \inf_{x \in [0,1]} \sup \{|y| : y \in [((E_x - x) \cap [0,a_n])\cup((E_x - x) \cap [-a_n,0])]\}.$$

Then $0 \leq a_{n+1} < a_n$

We claim $a_n > 0$ for all $n$. If not, there exists a natural number $n_o$ so that $a_{n_o} = 0$, but $a_{(n_o-1)} > 0$.

$$a_{n_o} = \tfrac{1}{2} \inf_{x \in [0,1]} \sup\{|y| : y \in [(E_x - x) \cap [0,a_{(n_o-1)})]]\cup[(E_x - x) \cap [-a_{(n_o-1)},0]]\}.$$

Since zero is an accumulation point of $(E_x - x)$,
$[(E_x - x) \cap [0,a_{(n_o-1)})]]\cup[(E_x - x) \cap [-a_{(n_o-1)},0]] \neq \emptyset$. The number

$$r_x = \sup \{|y| : y \in [(E_x - x) \cap [0,a_{(n_o-1)})]]\cup[(E_x - x) \cap [-a_{(n_o-1)},0]]\}$$

is positive for every $x \in [0,1]$ and $\inf_{x \in [0,1]} \{r_x\} = 0$. Thus there is a sequence $\{x_n\}_{n=1}^{\infty}$ so that $\lim_{n\to\infty} r_{x_n} = 0$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded, without loss of generality we can assume that $x_n$ converges to $x_o$. From the continuity of $E$ follows that $E_{x_n}$ tends to $E_{x_o}$

This implies $\lim_{n\to\infty} r_{x_n} = r_{x_o} = 0$. But $E_{x_o}$ is the path leading to $x_o$ and hence $r_{x_o}$ cannot be zero. So $a_n > 0$ for all n. Also

$$a_n < \tfrac{1}{2}\, a_{n-1} < \tfrac{1}{2}\cdot\tfrac{1}{2}\, a_{n-2} < \ldots < (\tfrac{1}{2})^n a_1 = (\tfrac{1}{2})^n$$

Thus $\lim_{n\to\infty} a_n < \lim_{n\to\infty} (\tfrac{1}{2})^n = 0$ implying $a_n$ decreases to

zero. It is clear that $(E_x \cap [x + a_{n+1}, x+a_{n-1}] \cup (E_x \cap [x-a_{n-1}, x-a_{n+1}]) \neq \emptyset$ for all $x \in [0,1]$. The sequence $\{a_n\}_{n=1}^{\infty}$ has all the desired properties.

Lemma 7: Let $\{g_n(x)\}_{n=1}^{\infty}$, $\{h_n(x)\}_{n=1}^{\infty}$ be sequences of lower, upper semi Borel functions of the class $\alpha$, respectively. If

a) $f(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x)$,

then $f \in B_{\alpha+1}$.

b) $f(x) = \limsup_{n\to\infty} g_n(x)$,

then $f \in B_{\alpha+2}$.

Proof: (a) Suppose $f(x) = \lim_{n\to\infty} h_n(x) = \sup_n \inf_{m\geq n} h_m(x)$.

The set

$$\{x\colon f(x) \leq t\} = \bigcap_{n=1}^{\infty} \{x\colon \inf_{m\geq n} h_m(x) \leq t\}$$

$$= \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \{x\colon \inf_{m\geq n} h_m(x) < t + \tfrac{1}{k}\}$$

$$= \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=n}^{\infty} \{x: h_m(x) < t + \tfrac{1}{k}\}, \text{ and the set}$$

$$\{x: f(x) \geq t\} = \{x: \lim_{n\to\infty} g_n(x) \geq t\} = \{x: \inf_n \sup_{m\geq n} g_m(x) \geq t\}$$

$$= \bigcap_{n=1}^{\infty} \{x: \sup_{m\geq n} g_m(x) \geq t\}$$

$$= \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \{x: \sup_{m\geq n} g_m(x) > t - \tfrac{1}{k}\}$$

$$= \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=n}^{\infty} \{x: g_m(x) > t - \tfrac{1}{k}\}.$$

Since $g_m(x)$ and $h_m(x)$ are lower, upper semi-Borel function of the class $\alpha$, respectively, the sets $\{x: h_m(x) < t + 1/k\}$, and $\{x: g_m(x) > t - 1/k\}$ are of the additive Borel class $\alpha$ for all t and k. Hence the sets $\{x: f(x) \leq t\}$ and $\{x: f(x) \geq t\}$ are of multiplicative class $\alpha+1$ for all $t \in R$. Thus $f \in B_{\alpha+1}$.

b) If $f(x) = \lim\sup g_n(x)$, then we have

$$\{x: f(x) \leq t\}^{\sim} = \{x: f(x) > t\} = \bigcup_{r=1}^{\infty} \{x: f(x) \geq t + \tfrac{1}{r}\}$$

$$= \bigcup_{r=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=n}^{\infty} \{x: g_m(x) > t - \tfrac{1}{k} + \tfrac{1}{r}\}.$$

Hence the sets $\{x: f(x) \leq t\}$ and $\{x: f(x) \geq t\}$ are of the multiplicative Borel class $\alpha+2$. Thus $f \in B_{\alpha+2}$.

<u>Lemma 8</u>: Let $E = \{E_x: x \in [0,1]\}$ be a continuous system of paths, $F(x)$ a continuous function defined on $[0,1]$, and $\{a_n\}_{n=1}^{\infty}$ be the corresponding sequence as in lemma 6. Then the function $\bar{F}_n(x) = \sup\{[F(y)-F(x)]/(y-x): y \in I_n(x)\}$, where $I_n(x) = (E_x \cap [x+a_{n+1}, x+a_{n-1}]) \cup (E_x \cap [x-a_{n-1}, x-a_{n+1}])$ is an upper semi continuous function.

<u>Proof</u>: On the contrary suppose that $\bar{F}_n(x)$ is not an upper semi continuous function. Then there is a point $x_0$ and a positive $\varepsilon_0$ so that for all $\delta = 1/m$ a point $y_m$ could be found such that $|y_m - x_0| < 1/m$, and $\bar{F}_n(y_m) > \bar{F}_n(x_0) + \varepsilon_0$, where $\bar{F}_n(y_m) = \sup\{[F(y)-F(y_m)]/(y-y_m): y \in I_n(y_m)\}$. Since $F(x)$

is a continuous function, without loss of generality we can assume each $E_x$ is a closed set.

For each m the function $\dfrac{F(y)-F(y_m)}{y-y_m}$ is a continuous function and therefore attains its maximum on the closed set $I_n(y_m)$.

So there is a $z_m \in I_n(y_m)$ such that $\bar{F}_n(y_m) = [F(z_m)-F(y_m)]/(z_m-y_m)$. The sequence $\{z_m\}_{m=1}^{\infty}$ is a bounded sequence, so it has a subsequence $\{z_{m_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} z_{m_k} = z_0$. We have

$$\lim_{k\to\infty} \bar{F}_n(y_{m_k}) = \lim_{k\to\infty} \frac{F(z_{m_k}) - F(y_{m_k})}{z_{m_k} - y_{m_k}} = \frac{F(z_0) - F(x_0)}{z_0 - x_0} \quad \text{, and}$$

$z_{m_k} \in I_n(y_{m_k})$ implying either $z_{m_k} \in E_{y_{m_k}} \cap [y_{m_k} + a_{n+1}, y_{m_k} + a_{n-1}]$ or $z_{m_k} \in E_{y_{m_k}} \cap [y_{m_k} - a_{n-1}, y_{m_k} - a_{n+1}]$. Since $y_{m_k} \to x_0$ and $z_{m_k} \to z_0$, by the continuity of the system of paths $E$, the sequence of sets $E_{y_{m_k}} \cap [y_{m_k} + a_{n+1}, y_{m_k} + a_{n-1}]$ tends to $E_{x_0} \cap [x_0 + a_{n+1}, x_0 + a_{n-1}]$, and $E_{y_{m_k}} \cap [y_{m_k} - a_{n-1}, y_{m_k} - a_{n+1}]$ tends to $E_{x_0} \cap [x_0 - a_{n+1}, x_0 - a_{n-1}]$. If infinitely many of $z_{m_k}$'s are in $E_{y_{m_k}} \cap [y_{m_k} + a_{n+1}, y_{m_k} + a_{n-1}]$ for $k=1,2,3,\ldots$, then $z_0 \in E_{x_0} \cap [x_0 + a_{n+1}, x_0 + a_{n-1}]$ and $\lim_{k\to\infty} \bar{F}_n(y_{m_k}) =$

$$\frac{F(z_0)-F(x_0)}{z_0-x_0} \leq \bar{F}_n(x_0).$$ But $\bar{F}_n(y_{m_k}) > \bar{F}_n(x_0) + \varepsilon_0$ implying $\bar{F}_n(x_0) + \varepsilon_0 \leq \bar{F}_n(x_0)$ which is impossible.

Similarly if infinitely many of $z_{m_k}$ are in $E_{y_{m_k}} \cap [y_{m_k} - a_{n-1}, y_{m_k} - a_{n+1}]$, then $\bar{F}_n(x_0) + \varepsilon_0 \leq \bar{F}_n(x_0)$ which is impossible. So the function $\bar{F}_n(x)$ is an upper semi continuous function.

<u>Lemma 9</u>: Let $\Xi = \{E_x: x \in [0,1]$ be a continuous system of paths, $F(x)$ a continuous function defined on $[0,1]$, and $\{a_n\}_{n=1}^{\infty}$ the corresponding sequence as in lemma 6. Then the function

$$\underline{F_n}(x) = \sup \{\frac{F(y)-F(x)}{y-x}: y \in E_x \cap [(x+a_{n+1}, x+a_{n-1}) \cup (x-a_{n-1}, x-a_{n+1})]\}$$

is a lower semi continuous function.

<u>Proof</u>: Let $x_0 \in [0,1]$, and $\underline{F_n}(x_0) = \dfrac{F(t_0)-F(x_0)}{t_0 - x_0}$. Then there are two possibilities for $t_0$:

1) $t_0 \in E_{x_0} \cap [(x_0 + a_{n+1}, x_0 + a_{n-1}) \cup (x_0 - a_{n-1}, x_0 - a_{n+1})]$

2) $t_0 \in \{x_0 + a_{n+1}, x_0 + a_{n-1}, x_0 - a_{n-1}, x_0 - a_{n+1}\}$.

If $\underline{F_n}(x_0) = 0$ let $G(x) = F(x) + cx$ where $c \neq 0$, then $\underline{F_n}(x_0) = \underline{G_n}(x_0) + c$ implying $\underline{G_n}(x_0) \neq 0$.

Thus without loss of generality we can assume that $\underline{F_n}(x_0) \neq 0$.

<u>Case 1</u>:

Suppose $t_0 \in E_{x_0} \cap [(x_0 + a_{n+1}, x_0 + a_{n-1}) \cup (x_0 - a_{n-1}, x_0 - a_{n+1})]$.

Let $0 < \varepsilon < \frac{1}{4}|F(t_0) - F(x_0)|$.

Since $F(x)$ is a continuous function defined on $[0,1]$, for $\varepsilon > 0$ there is a positive $\gamma$ so that $|F(x_0)-F(y)| < \varepsilon$ when $|y-x_0| < \gamma$.

Let $\gamma_1 = \frac{1}{2} \min \{\varepsilon, \gamma, \frac{a_{n+1}}{2}, |t_0-x_0-a_{n+1}|, |t_0-x_0-a_{n-1}|, |t_0-x_0+a_{n+1}|, |t_0-x_0+a_{n-1}|\}$. Since $E: x \to E_x$ is a continuous function, there is positive $\delta$ less than $\gamma_1$ so that $d_H(E_{x_0}, E_y) < \gamma_1$ when $|y-x_0| < \delta$. For all $y$ so that $|y-x_0| < \delta$

there is a $s_y \in E_y$ so that $|s_y - t_0| < \gamma_1$. Therefore $|F(s_y)-F(t_0)| < \varepsilon$, and $|F(y)-F(x_0)| < \varepsilon$. We have the following inequalities:

(1) $F(x_0) - \varepsilon < F(y) < F(x_0) + \varepsilon$

(2) $F(t_0) - \varepsilon < F(s_y) < F(t_0) + \varepsilon$

(3) $x_0-\gamma_1 < x_0-\delta < y < x_0 + \delta < x_0 + \gamma_1$

(4) $t_0 - \gamma_1 < s_y < t_0 + \gamma_1$.

From (1) and (2) follows

(5) $F(t_0) - F(x_0) - 2\varepsilon < F(s_y) - F(y) < F(t_0) - F(x_0) + 2\varepsilon$.

From (3) and (4) follows

(6) $(t_0-x_0) - 2\gamma_1 < s_y - y < (t_0-x_0) + 2\gamma_1$.

From the inequalities (5) and (6) it follows that

if $s_y-y > 0$, then

(7) $$\frac{F(s_y)-F(y)}{s_y - y} > \frac{F(t_0)-F(x_0)-2\varepsilon}{(t_0-x_0) + 2\gamma_1},$$

if $s_y - y < 0$, and $F(t_0) - F(x_0) > 0$, then

$$(8) \quad \frac{F(s_y)-F(y)}{s_y - y} > \frac{F(t_0)-F(x_0) + 2\varepsilon}{(t_0 - x_0) + 2\gamma_1}, \quad \text{and}$$

if $s_y - y < 0$, and $F(t_0)-F(x_0) < 0$, then

$$(9) \quad \frac{F(s_y)-F(y)}{s_y - y} > \frac{F(t_0)-F(x_0) + 2\varepsilon}{(t_0 - x_0) - 2\gamma_1}$$

Hence in all three cases, since

$$\underline{F_n}(y)=\sup \left\{\frac{F(t)-F(y)}{t-y}: t \in E_y \cap [(y+a_{n+1}, y+a_{n-1}) \cup (y-a_{n-1}, y-a_{n+1})]\right\}$$

we have $\underline{F_n}(y) > \dfrac{F(t_0)-F(x_0) \pm 2\varepsilon}{(t_0 - x_0) \pm 2\gamma}$. Therefore

$$\liminf_{y \to x_0} \underline{F_n}(y) \geq \frac{F(t_0)-F(x_0)}{t_0 - x_0} = \underline{F_n}(x_0).$$

Case 2: Suppose $t_0 \in \{x_0+a_{n+1}, x_0+a_{n-1}, x_0-a_{n+1}, x_0-a_{n-1}\}$. In this case there is a sequence $\{t_m\}_{m=1}^{\infty}$, $t_m \in E_{x_0} \cap [(x_0+a_{n+1}, x_0+a_{n-1}) \cup$

$(x_0-a_{n-1}, x_0-a_{n+1})]$, so that $\lim_{m\to\infty} t_m = t_0$.

Let $o < \varepsilon < \dfrac{|F(t_0)-F(x_0)|}{4}$.

Since $F(x)$ is a continuous function defined on $[0,1]$, for $\varepsilon > 0$ there is a positive $\gamma$ so that $|F(x_0)-F(y)| < \varepsilon$ when

$|y-x_0| < \gamma$. Choose $t_0' \in \{t_m\}_{m=1}^{\infty}$ so that $|t_0'-t_0| < \gamma/2$, and let $\gamma_1 = \frac{1}{2} \min \{\varepsilon, a_{n+1}/4, |t_0'-x_0-a_{n+1}|, |t_0'-x_0-a_{n-1}|, |t_0'-x_0+a_{n+1}|, |t_0'-x_0+a_{n+1}|\}$. Since $E: x \to E_x$ is a continuous function, there is a positive $\delta$ less than $\min(\gamma_1, \gamma)$ so that $d_H(E_{x_0}, E_y) < \gamma_1$ when $|y-x_0| < \delta$.

For all $y$ so that $|y-x_0| < \delta$ there is a $s_y \in E_y \cap [(y+a_{n+1}, y+a_{n-1}) \cup (y-a_{n-1}, y-a_{n+1})] \cap (t_0'-\gamma_1, t_0'+\gamma_1)$, so $|F(s_y)-F(t_0')| < \varepsilon$. Since $|t_0'-t_0| < \gamma/2$, we have $|F(t_0')-F(t_0)| < \varepsilon$. Therefore $|F(s_y)-F(t_0)| < 2\varepsilon$, $|s_y-t_0| < 2\gamma$, and $|F(y)-F(x_0)| < \varepsilon$. We have the following inequalities

(10) $F(t_0)-F(x_0) - 3\varepsilon < F(s_y)-F(y) < F(t_0) - F(x_0) + 3\varepsilon$,

(11) $(t_0-x_0) - 3\gamma < s_y-y < (t_0-x_0) + 3\gamma$.

From the inequalities (10) and (11) similarly as in case 1 follows that:

$$\underline{F_n}(y) \geq \frac{F(s_y) - F(y)}{s_y - y} > \frac{F(t_0) - F(x_0) \pm 3\varepsilon}{t_0 - x_0 \pm 3\gamma}.$$ Hence

$$\liminf_{y \to x_0} \underline{F_n}(y) \geq \frac{F(t_0) - F(x_0)}{t_0 - x_0} = \underline{F_n}(x_0).$$

So $\underline{F_n}(x)$ is a lower semi continuous function.

Proof of Theorem 5:

$\overline{F}_n(x)$, $\underline{F_n}(x)$ are upper, lower semi continuous functions respectively. Since

$E_x \cap [(x+a_{n+1}, x+a_{n-1}) \cup (x-a_{n-1}, x-a_{n+1})] \cap [(x+a_{n+2}, x+a_n) \cup (x-a_n, x-a_{n+2})]$

in not empty, points lost by $\underline{F_n}(x)$ are picked up by $\underline{F_{n+1}}(x)$. Let

$$g_n(x) = \min (\overline{F}_n(x), \overline{F}_{n+1}(x), \overline{F}_{n-1}(x)),$$
$$h_n(x) = \max (\underline{F_n}(x), \underline{F_{n+1}}(x), \underline{F_{n-1}}(x)).$$

Then $g_n(x)$, $h_n(x)$ are upper, lower semi continuous functions respectively and $g_n(x) \leq \overline{F}_n(x) \leq h_n(x)$. By theorem 11 on page 155 of [10] for each natural number n there is a continuous function $P_n(x)$ so that $g_n(x) \leq P_n(x) \leq h_n(x)$. Since $\overline{F}'_E(x) = \limsup_{n\to\infty} \underline{F_n}(x) = \limsup_{n\to\infty} F_n(x)$,

$$\overline{F}'_E(x) = \limsup_{n \to \infty} h_n(x) = \limsup_{n \to \infty} g_n(x) = \limsup_{n \to \infty} P_n(x).$$

So $\overline{F}'_E(x)$ is a function in Baire class 2. If F is E-differentiable, then $F'_E(x) = \lim_{n\to\infty} \underline{F_n}(x) = \lim_{n\to\infty} \overline{F}_n(x) = \lim_{n\to\infty} P_n(x)$. Thus $F'_E(x)$ is a function in Baire class 1.

Remark: We could have also used lemma 7 to prove theorem 5 since $\overline{F}'_E(x) = \limsup_{n\to\infty} g_n(x) = \limsup_{n\to\infty} h_n(x)$.

Corollary 10: The congruent derivative (if it exists everywhere) and the extreme congruent derivative of a continuous function are in Baire class one and Baire class two respectively.

Theorem 11: Let $E = \{E_x : x \in [0,1]\}$ be a bilateral system of paths, $\{a_n\}_{n=1}^{\infty}$ be a positive sequence with the following properties:

(i) $\{a_n\}$ decreases to zero;

(ii) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$;

(iii) the sets $E_x \cap [x+a_{n+1}, x+a_n]$ and $E_x \cap [x-a_n, x-a_{n+1}]$ are both nonempty for all $x \in [0,1]$.

Also suppose that F is a real function so that for each natural number n.

$\phi_n(x,a_{n+1},a_n) = \sup\{F(y)-F(x): y \in E_x \cap [x+a_{n+1}, x+a_n]\}$ and

$\psi_n(x,a_{n+1},a_n) = \inf\{F(y)-F(x): y \in E_x \cap [x-a_n, x-a_{n+1}]\}$ are

functions of Baire class $\alpha$. Then $\overline{F}'_E$ is a function of Baire class $\alpha+2$.

Proof: Define $\phi(x,a,b) = \sup \{\frac{F(y)-F(x)}{y-x} : y \in E_x \cap [x+a,x+b]\}$.

$F(y)-F(x) \geq \psi_n(x,a_{n+1},a_n)$, and $-a_n < y - x < -a_{n+1}$ when

$y \in E_x \cap [x-a_n, x-a_{n+1}]$. So

(11.1) $\frac{F(y)-F(x)}{y-x} \leq \frac{\psi_n(x,a_{n+1},a_n)}{y-x} \leq \frac{\psi_n(x,a_{n+1},a_n)}{-a_n}$ when $\psi_n(x,a_{n+1},a_n) > 0$,

and

(11.2) $\frac{F(y)-F(x)}{y-x} \leq \frac{\psi_n(x,a_{n+1},a_n)}{y-x} \leq \frac{\psi_n(x,a_{n+1},a_n)}{-a_{n+1}}$ when $\psi_n(x,a_{n+1},a_n) \leq 0$.

Therefore

(11.3) $\Phi\ (x,-a_n,-a_{n+1}) \leq \frac{\psi_n(x,a_{n+1},a_n)}{-a_n}$ when $\psi_n(x,a_{n+1},a_n) > 0$, and

(11.4) $\Phi\ (x,-a_n,-a_{n+1}) \leq \frac{\psi_n(x,a_{n+1},a_n)}{-a_{n+1}}$ when $\psi_n(x,a_{n+1},a_n) \leq 0$.

On the other hand when $y \in E_x \cap [x-a_n, x-a_{n+1}]$ we have

(11.5) $\frac{F(y)-F(x)}{y-x} \leq \Phi\ (x,-a_n,-a_{n+1})$.

So

(11.6) $F(y)-F(x) \geq (y-x)\ \Phi(x,-a_n,-a_{n+1}) \geq -a_n\ \Phi(x,-a_n,-a_{n+1})$ when $\Phi(x,-a_n,-a_{n+1}) > 0$ and

(11.7) $F(y)-F(x) \geq (y-x)\cdot \Phi(x,-a_n,-a_{n+1}) \geq -a_{n+1}\ \Phi(x,-a_n,-a_{n+1})$ when $\Phi(x,-a_n,-a_{n+1}) \leq 0$.

Therefore

(11.8) $\frac{\psi_n(x,a_{n+1},a_n)}{-a_n} \leq \Phi(x,-a_n,-a_{n+1})$ when $\Phi(x,-a_n,-a_{n+1}) \geq 0$, and

$$(11.9)\qquad \frac{\psi_n(x,a_{n+1},a_n)}{-a_{n+1}} \le \Phi(x,-a_n,-a_{n+1}) \text{ when } \Phi(x,-a_n,-a_{n+1}) < 0.$$

So by combining (11.3), (11.4), (11.8), (11.9) and the fact that $\psi_n(x,a_{n+1},a_n)$ and $\Phi(x,-a_n,-a_{n+1})$ have opposite signs, we have

$$(11.10)\qquad \frac{\psi_n(x,a_{n+1},a_n)}{-a_{n+1}} \le \Phi(x,-a_n,-a_{n+1}) \le \frac{\psi_n(x,a_{n+1},a_n)}{-a_n}$$

when $\Phi(x,-a_n,-a_{n+1}) < 0$, and

$$(11.11)\qquad \frac{\psi_n(x,a_{n+1},a_n)}{-a_n} \le \Phi(x,-a_n,-a_{n+1}) \le \frac{\psi_n(x,a_{n+1},a_n)}{-a_{n+1}}$$

when $\Phi(x,-a_n,-a_{n+1}) \ge 0$.

Since $\limsup_{n\to\infty} \frac{\psi_n(x,a_{n+1},a_n)}{-a_{n+1}} = \limsup_{n\to\infty} \frac{a_n}{a_{n+1}} \cdot \frac{\psi_n(x,a_{n+1},a_n)}{-a_n}$

$$= \limsup_{n\to\infty} \frac{\psi_n(x,a_{n+1},a_n)}{-a_n},$$

we have

$$(11.12)\qquad \limsup_{n\to\infty} \Phi(x,-a_n,-a_{n+1}) = \limsup_{n\to\infty} \frac{\psi_n(x,a_{n+1},a_n)}{-a_n}.$$

Similarly when $y \in E_x \cap [x+a_{n+1},x+a_n]$

$$(11.13)\quad \frac{F(y)-F(x)}{y-x} \leq \frac{\phi_n(x,a_{n+1},a_n)}{a_{n+1}} \text{ when } \phi_n(x,a_{n+1},a_n) \geq 0 \text{ and}$$

$$(11.14)\quad \frac{F(y)-F(x)}{y-x} \leq \frac{\phi_n(x,a_{n+1},a_n)}{a_n} \text{ when } \phi_n(x,a_{n+1},a_n) < 0. \quad \text{So}$$

$$(11.15)\quad \Phi(x,a_{n+1},a_n) \leq \frac{\phi_n(x,a_{n+1},a_n)}{a_{n+1}} \text{ when } \phi_n(x,a_{n+1},a_n) > 0.$$

and

$$(11.16)\quad \Phi(x,a_{n+1},a_n) \leq \frac{\phi_n(x,a_{n+1},a_n)}{a_n} \text{ when } \phi_n(x,a_{n+1},a_n) \leq 0.$$

On the other hand

$\frac{F(y)-F(x)}{y-x} \leq \Phi(x,a_{n+1},a_n)$. Hence $F(y)-F(x) \leq (y-x)\ \Phi(x,a_{n+1},a_n) \leq$ $a_n \cdot \Phi(x,a_{n+1},a_n)$ when $\Phi(x,a_{n+1},a_n) \geq 0$, and
$F(y)-F(x) \leq (y-x)\ \Phi(x,a_{n+1},a_n) \leq a_{n+1}\ \Phi(x,a_{n+1},a_n)$
when $\Phi(x,a_{n+1},a_n) < 0$. So we have

$$(11.17)\quad \frac{\phi_n(x,a_{n+1},a_n)}{a_n} \leq \Phi(x,a_{n+1},a_n) \text{ when } \Phi(x,a_{n+1},a_n) \geq 0$$

and

$$(11.18)\quad \frac{\phi_n(x,a_{n+1},a_n)}{a_{n+1}} \leq \Phi(x,a_{n+1},a_n) \text{ when } \Phi(x,a_{n+1},a_n) \leq 0.$$

By combining (11.15), (11.16), (11.17), (11.18) and the fact that $\phi_n(x,a_{n+1},a_n)$ and $\Phi(x,a_{n+1},a_n)$ have the same signs we have

$$(11.19)\qquad \frac{\phi_n(x,a_{n+1},a_n)}{a_n} \le \Phi(x,a_{n+1},a_n) \le \frac{\phi_n(x,a_{n+1},a_n)}{a_{n+1}}$$

when $\Phi(x,a_{n+1},a_n) \ge 0$ and

$$(11.20)\qquad \frac{\phi_n(x,a_{n+1},a_n)}{a_{n+1}} \le \Phi(x,a_{n+1},a_n) \le \frac{\phi_n(x,a_{n+1},a_n)}{a_n}$$

when $\Phi(x,a_{n+1},a_n) <0$. Since

$$\limsup_{n\to\infty} \frac{\phi_n(x,a_{n+1},a_n)}{a_{n+1}} = \limsup_{n\to\infty} \frac{a_n}{a_{n+1}} \cdot \frac{\phi_n(x,a_{n+1},a_n)}{a_n}$$

$$= \limsup_{n\to\infty} \frac{\phi_n(x,a_{n+1},a_n)}{a_n}$$

we have

$$(11.21)\qquad \limsup_{n\to\infty} \Phi(x,a_{n+1},a_n) = \limsup_{n\to\infty} \frac{\phi_n(x,a_{n+1},a_n)}{a_n}$$

Therefore $\limsup_{n\to\infty} \Phi(x,a_{n+1},a_n)$ and $\limsup_{n\to\infty} \Phi(x,-a_n,-a_{n+1})$ are functions of Baire class $\alpha+2$, and

$\bar{F}_E'(x) = \max(\limsup_{n\to\infty} \Phi(x,a_{n+1},a_n), \limsup_{n\to\infty} \Phi(x,-a_n,-a_{n+1}))$. So $\bar{F}_E'(x)$

is a function of Baire class $\alpha+2$.

The following lemma shows that when E is a continuous system of paths and $F \in B_1$ the functions $\phi_n(x,a_{n+1},a_n)$ and $\psi_n(x,a_{n+1},a_n)$ are in $B_2$.

Lemma 12: Let F be a function of Baire class one defined on [0,1], $E = \{E_x: x \in [0,1]\}$ be a bilateral continuous system of paths and $\{a_n\}_{n=1}^{\infty}$ be the sequence of lemma 6. Then for each natural number n the functions

$\phi(x,a_{n+1},a_n) = \sup\{F(t): t \in E_x \cap [x+a_{n+1},x+a_n]\}$

and $\psi(x,a_n,a_{n+1}) = \inf\{F(t): t \in E_x \cap [x-a_n,x-a_{n+1}]\}$ are functions of Baire class two.

Proof: Let $r \in R$. Then

$\{x: \phi(x,a_{n+1},a_n) \leq r\} = \{x: \text{for } t \in E_x \cap [x+a_{n+1},x+a_n],$
$F(t) \leq r\} = \{x: E_x \cap [x+a_{n+1},x+a_n] \subset \{t: F(t) \leq r\}\}$ and
$\{x: \psi(x,a_n,a_{n+1}) \geq r\} = \{x: \text{for } t \in E_x \cap [x-a_n,x-a_{n+1}],$
$F(t) \geq r\} = \{x: E_x \cap [x-a_n,x-a_{n+1}] \subset \{t: F(t) \geq r\}\}$.

Since $F \in B_1$, the sets $\{t: F(t) \leq r\}$ and $\{t: F(t) \geq r\}$ are sets of type $G_\delta$. Let $\{t: F(t) \leq r\} = \bigcap_{n=1}^{\infty} O_n$ and $\{t: F(t) \geq r\} = \bigcap_{k=1}^{\infty} G_k$ where for each n the sets $O_n$ and $G_n$ are open. We show that for every open set $O_n$, the set $\{x: E_x \cap [x+a_{n+1}, x+a_n] \subset O_n\}$ is open. Let $\{z_k\}_{k=1}^{\infty} \subset \widetilde{\{x: (E_x \cap [x+a_{n+1}, x+a_n]) \subset O_n\}}$. Then for each k there is a $t_k \in E_{z_k} \cap [z_k+a_{n+1}, z_k+a_n]$ and $t_k \notin O_n$. Let $\lim_{k\to\infty} z_k = z_o$ and $\lim_{k\to\infty} t_k = t_o$.

Since $E = \{E_x: x \in [0,1]\}$ is a continuous system of paths and $O_n$ is open, we have $t_o \in E_{z_o} \cap [z_o+a_{n+1}, z_o+a_n]$ and $t_o \notin O_n$. Hence $z_o \in \widetilde{\{x: (E_x \cap [x+a_{n+1}, x+a_n]) \subset O_n\}}$ implies that the set $\{x: (E_x \cap [x+a_{n+1}, x+a_n])) \subset O_n\}$ is an open set. Similarly the set $\{x: (E_x \cap [x-a_n, x-a_{n+1}]) \subset G_k\}$ is an open set. So

$$\{x: \phi(x, a_{n+1}, a_n) \le r\} = \{x: (E_x \cap [x+a_{n+1}, x+a_n]) \subset \bigcap_{m=1}^{\infty} O_m\}$$

$$= \bigcap_{m=1}^{\infty} \{x: (E_x \cap [x+a_{n+1}, x+a_n]) \subset O_m\}$$ is a $G_\delta$ set. Also

$$\{x: \psi(x, a_n, a_{n+1}) \ge r\} = \{x: (E_x \cap [x-a_n, x-a_{n+1}]) \subset \bigcap_{k=1}^{\infty} G_k\}$$

$$= \bigcap_{k=1}^{\infty} \{x: (E_x \cap [x-a_n, x-a_{n+1}]) \subset G_k\}$$ is a $G_\delta$ set. Therefore the

functions $\phi(x, a_{n+1}, a_n)$ and $\psi(x, a_n, a_{n+1})$ are functions of Baire class two.

Corollary 13: Let $E = \{E_x: x \in [0,1]\}$ be a bilateral continuous system of paths and F a function of Baire class one defined on [0,1]. If $\{a_n\}_{n=1}^{\infty}$ is a positive sequence such that it decreases to zero, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ and for all $x, E_x \cap [x + a_{n+1}, x + a_n] \neq \emptyset$, $E_x \cap [x - a_n, x - a_{n+1}] \neq \emptyset$. Then $\overline{F}'_E(x)$ is a function of Baire class four.

Proof: By Lemma 12 the functions $\phi_n(x,a_{n+1},a_n) = \phi(x,a_{n+1},a_n)-F(x)$ and $\psi_n(x,a_{n+1},a_n) = \psi(x,a_{n+1},a_n)-F(x)$ are functions of Baire class two. Thus by Theorem 11 $\overline{F}'_E(x)$ is a function of Baire class four.

Even for a continuous system of paths, the extreme path derivatives of a function in Baire class two are not necessarily Borel functions. M. Laczkovich in [ 6 ] showed that there exists a function in Baire class two and a closed set P having zero as a point of accumulation so that the congruent extreme derivative of F(x) is not a Borel measurable function.

Lemma 14: There is a Borel set A, and a perfect set P such that $\{x: x + P \subset A\}$ is not Borel.

Proof: Let $F = \{0.\ a_1a_2a_3\ldots: a_i = 0 \text{ or } 1\}$ and

$P = \{0.\ b_1b_2b_3\ldots: b_i = 2 \text{ or } 4\}$.

If $x_1+y_1 = x_2+y_2$, $x_1$ and $x_2$ elements of F, $y_1$ and $y_2$ are elements of P, then $x_1 = x_2$ and $y_1 = y_2$.

So there exists a $G_\delta$ set $G \subset F \times P$ such that the set $C = \{x: \exists\, y \text{ with } (x,y) \in G\}$ is not a Borel set. Let $A = \{x+y: x \in F, y \in P\} \setminus \{(x+y): (x,y) \in G\}$. The set $\{x+y: x \in F, y \in P\}$ is a closed set and the set $\{x+y: (x,y) \in G\} = B$ is a $G_\delta$ set since $\phi(x,y) = x + y$ is a one-one and continuous function on $F \times P$, hence $\phi$ is a homeomorphism and $B = \phi(G)$. So $A$ is an $F_\sigma$ set. The set

$$\{x: x+P \subset A\} \cap F = \{x \in F: \forall\, y \in P, x + y \in A\}$$
$$= \{x \in F: \forall\, y \in P, (x,y) \notin G\} = F \setminus C$$

which is not a Borel set. Hence $\{x: x+P \subset A\}$ is not a Borel set.

<u>Example 15</u>: There is a function $F \in B_2$ and a perfect set $P$ with $0 \in P$ so that the congruent extreme derivative of $F$ with respect to $P$, i.e. $\bar{F}'_P(x)$ is not a Borel function.

<u>Proof</u>: Let $F = \{0.a_1a_2a_3\ldots: a_i = 0 \text{ or } 1\}$, $P = \{0.b_1b_2b_3\ldots: b_i = 2 \text{ or } 4\}$, $A$ be an analytic subset of $F$ which is not a Borel set, $\{y_n\}_{n=1}^{\infty}$ be a sequence in $P$ so that $\lim y_n = 0$, and $(y_{n+1}, y_n) \cap P \neq \emptyset$ for all natural number $n$. For each $n \in N$ choose a set $U_n \subset F \times [y_{n+1}, y_n]$ which is a $G_\delta$ set and $A = \{x: (x,y) \in U_n\}$. Let $U = \bigcup_{n=1}^{\infty} U_n$. Since all

the sets $U_n$ are disjoint, $U$ is a $G_\delta$ set. Let $B = \{(x+y): x \in F, y \in P$ and $(x,y) \notin U\}$. Then the set $B$ is an $F_\sigma$ set. So the function $F(x) = -\chi_B(x)$ is a function of Baire class two. If $x \in F\setminus A$, then for each $h \in P$ we have $(x,h) \notin U$. Thus $x + h \in B$, and $(x,0) \notin U$, so $x \in B$ and

$$F_P'(x) = \limsup_{\substack{h\to 0\\ h\in P}} \frac{F(x+h)-F(x)}{h} = \limsup_{\substack{h\to 0\\ h\in P}} \frac{-1+1}{h} = 0.$$

If $x \in A$, then there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset P$ so that $(x,z_n) \in U$ for all $n \in N$, and $\lim z_n = 0$. Therefore $x + z_n \notin B$ for all $n \in N$. In this case

$$F_P'(x) = \limsup_{\substack{h\to 0\\ h\in P}} \frac{F(x+h)-F(x)}{h} \geq \lim_{n\to\infty} \frac{F(x+z_n)-F(x)}{z_n} \geq \lim_{n\to\infty} \frac{0+1}{z_n} = +\infty .$$

So $\bar{F}_P'(x) = +\infty$.

Hence we have $\bar{F}_P'(x) = \begin{cases} 0 \text{ if } x \in F\setminus A \\ +\infty \text{ if } x \in A \end{cases}$ which is not a Borel measurable function.

As we show in the following theorem the extreme path derivatives of a Borel function with respect to a continuous system of paths are always a measurable function.

<u>Theorem 16</u>: If $E = \{E_x\colon x \in [0,1]\}$ is a continuous system of paths and $F(x)$ is a Borel measurable function defined on $[0,1]$. Then $\bar{F}'_E(x)$ is a Lebesgue measurable function.

<u>Proof</u>: Let $\{a_n\}_{n=1}^{\infty}$ be the sequence as in Lemma 6 and $r \in R$. Define $F_n(x) = \sup \{[F(y)-F(x)]/(y-x)\colon y \in E_x \cap ([x+a_{n+1},x+a_n] \cup [x-a_n,x-a_{n+1}])\}$. For simplicity let $I_n(x) = E_x \cap ([x+a_{n+1},x+a_n] \cup [x-a_n,x-a_{n+1}])\}$. The set $\{x\colon F_n(x) > r\} = \{x\colon \sup \{[F(y)-F(x)]/(y-x)\colon y \in I_n(x)\} > r\} = \{x\colon \exists\, y_0 \in I_n(x)$ so that $[F(y_0)-F(x)]/(y_0-x) > r\} = P_1(A)$ where

$$A = \{(x,y)\colon \frac{F(y)-F(x)}{y-x} > r\} \cap \Big( \bigcup_{x\in[0,1]} (\{x\} \times I_n(x))\Big).$$

The set $B = \bigcup_{x\in[0,1]} (\{x\} \times I_n(x))$ is a closed set in $R^2$ since if

$z_m = (x_m,y_m) \in B$, and $\lim_{m\to\infty} z_m = z_0 = (x_0,y_0)$, $y_m \in I_n(x_m) =$

$E_{x_m} \cap ([x_m+a_{n+1},x_m+a_n] \cup [x_m-a_n,x_m-a_{n+1}])$ and $\lim_{m\to\infty} y_m = y_0$,

$\lim_{m\to\infty} x_m = x_0$. Since $E = \{E_x\colon x \in [0,1]\}$ is a continuous system of

paths $y_0 \in I_n(x_0)$, $z_0 = (x_0,y_0) \in B$. Since $F$ is a Borel measurable function, the set $\{(x,y): \frac{F(y)-F(x)}{y-x} > r\}$ is a Borel measurable set. Thus the set A is a Borel measurable set. Therefore $\{x: F_n(x) > r\}$ is an analytic set, and hence is a measurable set. Therefore $F_n(x)$ is a measurable function and $\overline{F}'_E(x) = \limsup_{n\to\infty} F_n(x)$ must be a measurable function.

We now briefly discuss the Borel measurability (measurability) of path derivatives. Let $E = \{E_x: x \in [0,1]\}$ be a system of paths, and $\{a_n\}$ be a positive decreasing sequence such that $E_x \cap [[x-a_n,x-a_{n+1}] \cup [x+a_{n+1},x+a_n]] \neq \emptyset$ for all x. When F is a Borel measurable function, finding a Borel measurable selection for the family of sets $\{E_x \cap [[x-a_n,x-a_{n+1}] \cup [x+a_{n+1},x+a_n]]: x \in [0,1]\}$ guarantees the Borel measurability of $F'_E$.

The following lemma shows that we are able to find an upper semi continuous selection when the system of paths is continuous.

<u>Lemma 17</u>: Let $E = \{E_x: x \in [0,1]\}$ be a continuous system of paths and let $\{a_n\}_{n=1}^{\infty}$ be the sequence as in lemma 6. Then for every natural number n the family $\{E_x \cap ((x+a_{n+1}, x+a_{n-1}) \cup (x-a_{n-1}, x-a_{n+1})), x \in [0,1]\}$ has an upper semi continuous selection.

<u>Proof</u>: Let $P_n(x) = \sup\{E_x \cap ((x+a_{n+1}, x+a_{n-1}) \cup (x-a_{n-1}, x-a_{n+1}))\}$, we show that $P_n(x)$ is an U.S.C. function. There are two possibilities:

(1) $P_n(x) \in E_x \cap ((x+a_{n+1}, x+a_{n-1}) \cup (x-a_{n-1}, x-a_{n+1}))$;

(2) $P_n(x) \in \{x+a_{n-1}, x-a_{n+1}\}$.

If case 1 happens, let $0 < \varepsilon \in < \frac{1}{4}\min\{|x+a_{n-1}-P_n(x)|, |P_n(x) - x - a_{n+1}|, |P_n(x) - x + a_{n-1}|, |P_n(x) - x + a_{n+1}|\}$.

Since E is a continuous system of paths, there exists a positive $\gamma < \varepsilon$ so that $d_H(E_z, E_x) < \varepsilon$, when $|z-x| < \gamma$. Since $E_x$ is a closed subset of the real line, $P_n(x) \in E_x$ and for all z so that $|z-x| < \gamma$, there exist a $t_z \in E_z \cap (P_n(x) - \varepsilon, P_n(x) + \varepsilon) \subset E_z \cap ((z+a_{n+1}, z+a_{n-1}) \cup (z-a_{n-1}, z-a_{n+1}))$. So $P_n(x) - \varepsilon < t_z \leq P_n(z)$.

Hence for all z so that $|z-x| < \gamma$ we have $P_n(x) \leq P_n(z) + \varepsilon$. Then $P_n(x) \leq \liminf_{z\to x} P_n(z)$. So $P_n(x)$ is an upper semi continuous function. Suppose $P_n(x) \in \{x+a_{n-1}, x-a_{n+1}\}$, we treat the case $P_n(x) = x+a_{n-1}$. If $P_n(x) = x+a_{n-1}$,

there is a sequence $\{q_m\}_{m=1}^{\infty} \subset E_x \cap (x+a_{n+1}, x+a_{n-1})$ so that $\lim q_m = x+a_{n-1}$. For $\varepsilon > 0$ choose $n_o$ so large such that $x+a_{n-1} - q_{n_o} < \min(\varepsilon, \frac{1}{2}(a_n-a_{n+1}))$. Then for $0<\varepsilon'< \min(\varepsilon, \frac{1}{2}(x+a_{n-1}-q_{n_o}))$, there is a positive $\gamma < \varepsilon'$ such that $d_H(E_z, E_x) < \varepsilon'$ when $|z-x| < \gamma$.

Since $q_{n_o} \in E_x$, for all z such that $|z-x| < \gamma$ there exists a $t_z \in E_z \cap (q_{n_o} - \varepsilon', q_{n_o} + \varepsilon') \subset E_z \cap (z+a_{n-1}, z+a_{n+1})$

so $P_n(z) \geq t_z \geq q_{n_o} - \varepsilon' \geq q_{n_o} - \varepsilon > x + a_{n+1} = P_n(x)$.

Hence $\liminf_{z\to x} P_n(z) \geq P_n(x)$.

So $P_n(x)$ is an upper semi continuous function.

<u>Theorem 18</u>: Let $E = \{E_x: x \in [0,1]\}$ be a continuous system of paths, F be a function of Borel class $\alpha$ (measurable function) and suppose $F'_E(x)$ exist everywhere. Then $F'_E(x)$ is a function of semi Borel $\alpha+1$, hence it is a function of Borel class $\alpha+2$ (measurable function).

<u>Proof</u>: $P_n(x) = \sup \{E_x \cap ((x+a_{n+1}, x+a_{n-1}) \cup (x-a_{n-1}, x-a_{n+1}))\}$ is an upper semi continuous function and $P_n(x) \in E_x$ for all n.

$$F'_E(x) = \lim_{\substack{y\to x \\ y\in E_x}} \frac{F(y)-F(x)}{y-x} = \lim_{n\to\infty} \frac{F(P_n(x))-F(x)}{P_n(x)-x}$$

Since $P_n(x) - x > a_{n+1}$ or $P_n(x) - x < - a_{n+1}$, $P_n(x) - x \neq 0$ and the function $\frac{F(P_n(x))-F(x)}{P_n(x)-x}$ is a function of Borel class $\alpha+1$ (measurable) and therefore $F'_E \in B_{\alpha+2}$.

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