Generalizations of bounded variation for $1 \leq p < \infty$ and $k \geq 1$.

C.J.F. Upton

1. Definitions and Notations.

All functions to be considered will be defined on a finite interval [a,b] and may take real or complex values.

We denote by p any real number such that $1 \leq p < \infty$, and by k any positive integer.

If $a \leq x_0 < x_1 < .... < x_n \leq b$, the sequence (x_i), $i = 1,2,...,n$, will be called a subdivision of [a,b].

The first divided difference $[F(x_1) - F(x_0)]/(x_1 - x_0)$ is denoted by $Q_1(F; x_0, x_1)$ and the k-th order divided difference over the points $x_0, x_1, ..., x_k$ is defined inductively by

$$Q_k(F; x_0, x_1, ..., x_k) = (x_k - x_0)^{-1}[Q_{k-1}(F; x_1, ..., x_k) - Q_{k-1}(F; x_0, ..., x_{k-1})].$$

Then $Q_k(F; x_0, x_1, ..., x_k)$ may be written as

$$\sum_{i=0}^{k} \frac{F(x_i)}{\prod_{j \neq i} (x_i - x_j)},$$

see [5].

For $h > 0$, the first forward difference $F(x+h) - F(x)$ is denoted by $\Delta_h F(x)$ and, for all $k \geq 2$, $\Delta_h{}^k F(x)$ is defined inductively by $\Delta_h(\Delta_h{}^{k-1} F(x))$. It admits the identity

$$\Delta_h{}^k F(x) = k!\, h^k Q_k(F; x, x+h, ..., x+kh).$$

For any given variation V, the set of functions for which this variation is finite will be denoted by BV.

Unless otherwise stated, all of the variations to be considered will be formed by obtaining approximating sums on a typical subdivision of [a,b] and then finding the supremum over all possible subdivisions.

2. <u>Variations of First order</u>

Let

$$V_p(F) = V_p(F; a,b) = \sup \left(\Sigma_i |F(x_{i+1}) - F(x_i)|^p\right)^{1/p}.$$

For $p = 1$, this variation is classical. It was considered for $p = 2$ by N. Wiener [18] and for all $p > 1$ by L.C.Young and E.R.Love [3,4,19, 20]. It is a particular case of Φ-variation (L.C.Young [20]; see also J.Musielak and W.Orlicz [6]). When $p = 1$ it is a particular case of Λ-variation (D. Waterman [16]). When $p = 1$ it is also identical with the Riesz-variation given by

$$R_p(F) = R_p(F; a,b) = \sup \left(\Sigma_i |F(x_{i+1}) - F(x_i)|^p / (x_{i+1} - x_i)^{p-1}\right)^{1/p},$$

and for which F is BR_p if and only if F' exists almost everywhere and belongs to $L^p[a,b]$ (F.Riesz [8]).

3. <u>Variations of Second Order</u>

Let m be any strictly increasing function and let

$$SV_p(F) = SV_p(F; a,b) = \sup \left(\sum_i \left| \frac{F(x_{i+2}) - F(x_{i+1})}{m(x_{i+2}) - m(x_{i+1})} - \frac{F(x_{i+1}) - F(x_i)}{m(x_{i+1}) - m(x_i)} \right|^p \right)^{1/p}.$$

This variation is called the p-th slope variation with respect to m. It has been considered for $p = 1$ by F. N. Huggins [2], J.R.Webb [17] and A.M.Russell [10], and for $p \geq 1$ by V. Postelica [7]. The last has shown that F is BSV_p if and only if $F = \int f\, dm$, a Lebesgue-Stieltjes integral, where f is BV_p and is equal almost everywhere to the derivative of F with respect to m.

Let $m(x) = x$. Then $SV_p(F)$ may be written in terms of divided differences as

$$V^*_{2,p}(F) = V^*_{2,p}(F;a,b) = \sup\,(\Sigma_i\,|Q_1(F;x_{i+1},x_{i+2}) - Q_1(F;x_i,x_{i+1})|^p)^{1/p}.$$

This variation was considered for $p = 1$ by F.Riesz [9], and for $p \geq 1$ by A.M.Russell and C.J.F.Upton [13]. The latter authors showed, independently of Postelica, that F is $BV^*_{2,p}$ if and only if $F = \int^{\cdot} f$, where f is BV_p and $f = F'$ almost everywhere. They also showed that

$$V^*_{2,p}(F) \leq V_p(f) \leq 2^{(p-1)/p}\,V^*_{2,p}(F), \tag{1}$$

with best possible constants, and that, if $a < b < c$ and $F'(b)$ exists (a technical requirement), then there is no inequality which always exists between and $V^*_{2,p}(F;a,c)$ and $V^*_{2,p}(F;a,b) + V^*_{2,p}(F;b,c)$. (It was subsequently shown in [14] that

$$V^*_{2,p}(F;a,c) \leq 2^{(p-1)/p}\,\{V^*_{2,p}(F;a,b) + V^*_{2,p}(F;b,c)\}, \tag{2}$$

and that the constant $2^{(p-1)/p}$ is best possible.)

If the definition of $V^*_{2,p}(F;a,b)$ is varied by using in each term of an approximating sum the set of points $\{x_i, x'_i, x_{i+1}, x'_{i+1}\}$, where $x_i < x'_i < x_{i+1} < x'_{i+1}$, instead of using the set $\{x_i, x_{i+1}, x_{i+2}\}$, then a new definition of second order variation is obtained (see [14]), namely

$$V_{2,p}(F) = V_{2,p}(F;a,b) = \sup\,(\Sigma_i\,|Q(F;x_{i+1},x'_{i+1}) - Q_1(F;x_i,x'_i)|^p)^{1/p}.$$

As before, F is $BV_{2,p}$ if and only if $F = \int f$, where f is BV_p and $f = F'$ almost everywhere. But the inequality (1) can now be replaced by the equality

$$V_{2,p}(F) = V_p(f), \tag{3}$$

and the inequality (2) can be replaced by

$$V_{2,p}(F; a,c) \le V_{2,p}(F; a,b) + V_{2,p}(F; b,c).$$

Furthermore, the two variations are related by the inequalities

$$V^*_{2,p}(F) \le V_{2,p}(F) \le 2^{(p-1)/p} V^*_{2,p}(F), \tag{4}$$

with best possible constants. They are therefore equal when $p = 1$.

4. Variations of k-th order and p-th power.

The above variations $V^*_{2,p}(\,.\,)$ and $V_{2,p}(\,.\,)$ can be generalized as follows. Let $x_n < b$ and , for $i = 1,2,....,n$ and $j = 0,1,....., k-1$, let $x_{i,j}$ be such that $x_i = x_{i,0} < x_{i,1} <.......< x_{i,k-1} \le b$. Let

$$S = \sup\,(\Sigma_i\, |Q_{k-1}(F; x_{i+1,0},...., x_{i+1,k-1}) - Q_{k-1}(F; x_{i,0},...., x_{i,k-1})|^p)^{1/p},$$

where the supremum is taken over all possible sets $(x_{i,j})$. If the set $(x_{i,j})$ is restricted by the inequality $x_{i,k-1} < x_{i+1,0} = x_{i+1}$ for $i = 1,2,....,n-1$, then the two (k-1)-th order divided differences in a typical term of S are formed on two strictly non-overlapping sets of points and S is said to define the variation $V_{k,p}(F)$. If, however, the set $(x_{i,j})$ is restricted by the equation $x_{i,j} = x_{i+1,j-1}$ for $i = 0,1,..., n-1$ and $j = 1, 2,..., k-1$, then the two (k-1)-th order divided differences in a typical term of S are formed on two sets of points which overlap almost completely, and S is said to define the variation $V^*_{k,p}(F)$. The variation $V^*_{k,p}(F)$ was considered for $p = 1$ by A.M.Russell [11,12]. (See also G. Brown [1].)

As in the case when $k = 2$, if F is $BV_{k,p}$ (or $BV^*_{k,p}$), then F is a (k-1)-th integral of a function f, where f is BV_p and $f = F^{(k-1)}$ almost everywhere. Also

$$(k-1)!\, V_{k,p}(F) = V_p(f), \tag{5}$$

and
$$V_{k,p}(F) \le k^{(p-1)/p}\, V^*_{k,p}(F), \tag{6}$$

where the constant $k^{(p-1)/p}$ is again best possible. The equation (5) is the analogue of (3), and (6) is the analogue of the second inequality in (4). But the analogue of the first inequalty in (4) is not known.

5. <u>Variations with subintervals of equal length.</u>

It is convenient to consider variations in which some of the subintervals which are used are of equal length. These variations can be defined slightly differently. Let $h > 0$ and let

$$V_{k,p,h}(F) = \sup \left(\Sigma_i \left|h^{-k+1}\Delta_h^{k-1}[F(x_{i+1})-F(x_i)]\right|^p\right)^{1/p},$$

where $x_n+(k-1)h \le b$ and $x_i+(k-1)h < x_{i+1}$ for $i = 1,2,\ldots, n-1$, so that the expressions in a typical term are formed on two non-overlapping sets of points. Here the supremum is taken over all possible subdivisions (x_i) of $[a, b]$ and over all possible values of h.

Again, let

$$V^*_{k,p,h}(F) = \sup \left(\Sigma_i \left|h^{-k+1}\Delta_h^{k-1}[F(x_i+h)-F(x_i)]\right|^p\right)^{1/p},$$

where, now, $x_i + h = x_{i+1}$ for all i, so that the expressions in a typical term are formed on two sets of points which overlap almost completely. The supremum is taken as previously. For all functions in $BV_{k,p}$ the following relations can be shown.

$$V_{k,p,h}(F) = (k-1)!\, V_{k,p}(F),$$

$$V^*_{k,p,h}(F) \le (k-1)!\, V^*_{k,p}(F),$$

and
$$V^*_{k,p,h}(F) \le V_{k,p,h}(F).$$

Furthermore,

$$V_{k,p,h}(F) = V_p(f),$$

where f is BV_p and $f = F^{(k-1)}$ almost everywhere.

Proofs. Most of the proofs are analagous to those given in [14] and [15] for the case in which $p \geq 1$ and $k = 2$. They make use, where necessary, of results obtained in [11] when $p = 1$ and $k \geq 3$.

References.

1. Brown G. Continuous functions of bounded n-th variation. Proc. Edinburgh Math. Soc.(2) 16 (1969) 205-214.
2. Huggins, F.N. A generalization of a theorem of F. Riesz. Pacific J. Math. 39 (1971) 695-701.
3. Love, E.R. A generalization of absolute continuity. Journ. London Math.Soc. 26 (1951) 1-13.
4. Love, E.R & Young L.C. On fractional integration by parts. Proc. London Math. Soc. (2) 44 (1938) 1-35.
5. Milne-Thompson R.M. The calculus of finite differences. (Macmillan, London, 1933).
6. Musielak J. & Orlicz W. On generalized variations (I). Studia Math. 18 (1959),11-41.
7. Postolica V. Asupra primitivelor functiilor cu p-variatie marginita. St.Cerc.Mat. 33 (1981) 141-146.
8. Riesz F. Untersuchungen uber Systeme integrierbarer Funktionen. Math. Annalen 69 (1910) 449-497.
9. Riesz F. Sur certains systems singulieres d'equations integrales. Ann. l'Ecole Norm Sup.,Paris (3) 28 (1911) 33-68.
10. Russell, A.M. Functions of bounded second variation and Stieltjes-type integrals. J. London Math. Soc.(2) 2(1970) 193-208.
11. Russell, A.M. Functions of bounded k-th variation. Proc. London Math. Soc.(3) 26 (1973) 547-563.

12. Russell, A.M. Further results on an integral representation of functions of generalized variation. Bull.Austral. Math. Soc 18 (1978) 407-420.

13. Russell, A.M. & Upton C.J.F. A generalization of a theorem by F. Riesz. Analysis Math. 9 (1983) 69-77.

14. Upton C.J.F A generalization of second variation. Analysis Math. 11 (1985) 241-256.

15. Upton C.J.F Generalizations of third order variation. Melbourne University Mathematics Research Report No.6 (1985).

16. Waterman D. On convergence of Fourier series of functions of generalized variation. Studia Math. 44 (1972) 107-117.

17. Webb J.R. A Hellinger integral representation for bounded linear functionals. Pacific J. Math. 20 (1967) 327-337.

18. Wiener N. The quadratic variation of a function and its Fourier coefficients. J. Math. and Phys. (Massachusetts Inst.Tech.) 3 (1924) 73-94.

19. Young L.C. An inequality of the Holder type connected with Stieltjes integration. Acta Math. 76 (1936) 251-282.

20. Young L.C. Sur une generalisation de la notion de variation de puissance p-ieme bornee au sens de M.Wiener, et sur la convergence des series de Fourier. C.R. Acad. Sci. Paris 204 (1937) 470-472.

Mathematics Department,
University of Melbourne,
Parkville,
Victoria,
Australia.