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CONVERGENCE THEOREMS IN INTEGRATION THEORY

We shall define the Henstock integral [5, 6, 14], describe three convergence theorems [9, 11] and sketch one of the proofs.

A function f is said to be Henstock integrable to A on [a, b] if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that for any division (called δ-fine) given by

$$a = x_0 < x_1 < \ldots < x_n = b \text{ and } \xi_1, \xi_2, \ldots, \xi_n$$

satisfying $\xi_i - \delta(\xi_i) < x_{i-1} \leq \xi_i \leq x_i < \xi_i + \delta(\xi_i)$ for $i = 1, 2, \ldots, n$, we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right| < \varepsilon.$$

It is well-known that the Henstock integral is equivalent to the Denjoy integral [5, 7, 17] and to the Perron integral [5, 15, 17]. The Henstock integral is also known as the Kurzweil integral [8].

Next, we define major and minor functions and functions which are ACG_*. A function H is said to be a major function of a function f in [a, b] if

$$-\infty \neq \underline{D}H(x) \geq f(x) \quad \text{for every } x$$

where $\underline{D}$ denotes the lower derivative. A function G is said to be a minor function of f in [a, b] if $-G$ is a major function of $-f$ in [a, b].

A function F is said to be $AC_*(X)$ if for every $\varepsilon > 0$ there is $\eta > 0$ such that for every finite and infinite sequence of non-overlapping intervals

$\{[a_i, b_i]\}$ with $\Sigma_i\ |b_i - a_i| < \eta$ where a_i, $b_i \in X$ we have

$$\Sigma_i\ \omega(F; [a_i, b_i]) < \varepsilon,$$

where ω denotes the oscillation of F over $[a_i, b_i]$. Further, F is said to be ACG_* if [a, b] is the union of a sequence of closed sets X_i such that on each X_i the function F is $AC_*(X_i)$.

We consider the following conditions :

(i) the sequence f_n converges to f almost everywhere in [a, b] where each f_n is Henstock integrable on [a, b];

(ii) the primitives F_n of f_n converge uniformly on [a, b];

(iii) f_n have at least one common major function and at least one common minor function in [a, b];

(iv) the primitives F_n of f_n are ACG_* uniformly in n, that is, [a, b] is the union of a sequence of closed sets X_i such that on each X_i the functions F_n are $AC_*(X_i)$ uniformly in n; in other words, $\eta > 0$ in the definition of $AC_*(X_i)$ is independent of n;

(v) the primitives F_n of f_n satisfy the condition that [a, b] is the union of a sequence of closed sets X_i and for every i and $\varepsilon > 0$ there is an integer N such that for every partial division of [a, b] given by

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_p < b_p \leq b$$

with $a_1, b_1, a_2, b_2, \ldots, a_p, b_p \in X_i$ we have

$$\sum_{k=1}^{p} \omega(F_n - F_m; [a_k, b_k]) < \varepsilon \text{ whenever } n, m > N.$$

GENERALIZED DOMINATED CONVERGENCE THEOREM [11, 13]. If conditions (i), (ii) and (iii) hold, then f is Henstock integrable on [a, b] and

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx \quad \text{as } n \to \infty.$$

CONTROLLED CONVERGENCE THEOREM [9, 10, 12]. If conditions (i), (ii) and (iv) hold, then f is Henstock integrable on [a, b] and

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx \quad \text{as } n \to \infty.$$

GENERALIZED MEAN CONVERGENCE THEOREM [11]. If conditions (i), (ii) and (v) hold, then f is Henstock integrable on [a, b] and

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx \quad \text{as } n \to \infty.$$

As long as we can prove one of the above theorems, we can deduce from it the other two. For example, conditions (i), (ii) and (iii) imply (i), (ii) and (iv), which in turn imply (i), (ii) and (v) (see [11, 12]). Therefore it suffices to prove the mean convergence theorem. We sketch a proof as follows.

Suppose that conditions (i), (ii) and (v) hold. Let A be the limit of $F_n(b) - F_n(a)$ as $n \to \infty$. Given $\varepsilon > 0$, following the proof of the monotone convergence theorem [6] we can choose for each $x \in [a, b]$ an integer m, depending on ε and x, and a function $\delta(x) > 0$ such that for any δ-fine division we have

$$\begin{aligned}\left|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A\right| < &\left|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f_m(\xi_i)(x_i - x_{i-1})\right| \\ &+ \left|\sum_{i=1}^{n} f_m(\xi_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} \{F_m(x_i) - F_m(x_{i-1})\}\right| \\ &+ \left|\sum_{i=1}^{n} \{F_m(x_i) - F_m(x_{i-1})\} - A\right|\end{aligned}$$

in which m depends on ξ_i and such that each of the first two terms on the right side of the above inequality is small. The fact that the third term is also small follows from condition (v). Hence the proof of the generalized mean convergence theorem is complete.

We remark that Djvarsheishvili [2, 3] proved a convergence theorem involving the Denjoy integral, which is equivalent to the controlled convergence theorem. Other convergence theorems given in [4, 5, 6] and more will now follow as corollaries. Furthermore, these convergence theorems can be extended to more general integrals, for example, the Burkill approximately continuous integral [1, 18], and the Cesaro-Denjoy integral [16].

The controlled convergence theorem has been used by Chew to characterize the nonlinear functionals on the space of all Henstock integrable functions on [a, b].

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