Generalized Riemann Complete Integrals

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Definition. A tagged division of $[a,b]$ will be called a restricted tagged division of $[a,b]$ if it has the form

$$a = x_0 = z_1 < x_1 < z_2 < x_2 < z_3 < \cdots < x_{m-2} < z_{m-1} < x_{m-1} < z_m = x_m = b$$

where $x_0 = z_1$ is the tag of $[x_0,x_1]$, $x_m = z_m$ is the tag of $[x_{m-1},x_m]$ and z_j is the tag of both $[x_{j-1},z_j]$ and $[z_j,x_j]$ for $j = 2,3,\ldots,m-1$. If a restricted tagged division of $[a,b]$ has further the property that $z_j - x_{j-1} = x_j - z_j$, $j = 2,3,\ldots,m-1$, the division will be called a restricted symmetric tagged division of $[a,b]$.

It is clear that given $\delta(x) > 0$ defined on $[a,b]$ there exists a restricted tagged division of $[a,b]$ compatible with $\delta(x)$. That there exists a restricted symmetric tagged division of $[a,b]$ compatible with $\delta(x)$ follows from [2].

If f is a finite function defined on $[a,b]$, let two interval functions be defined by $F_\ell \equiv F_\ell(f,u,v) \equiv f(v)(v-u)$ and $F_r \equiv F_r(f,u,v) \equiv f(u)(v-u)$. It will be convenient to denote a pair of interval functions by a single letter in script face. For example we shall write $\mathcal{F}(u,v) = \{F_\ell(u,v), F_r(u,v)\}$ or, more briefly, $\mathcal{F} = (F_\ell, F_r)$.

Definition. The number I will be called the generalized Riemann complete (generalized symmetric Riemann complete) integral of f with respect to the pair of interval functions $\mathcal{h}(u,v) = \{h_\ell(u,v), h_r(u,v)\}$ on $[a,b]$ if, corresponding to $\epsilon > 0$, there is a function $\delta(x) > 0$ so that

$$|I-(\mathcal{D})\Sigma(h_\sigma+F_\sigma)| < \epsilon$$

for all restricted tagged divisions (restricted symmetric tagged divisions), $\mathcal{D}$ compatible with $\delta(x)$, where $\sigma = \ell$ or r, depeending on whether the tag of the interval is the right hand or left hand end point.

The notation for these integrals is

$$I = (GRC,h)\int_a^b f(t)dt \quad \text{and} \quad I = (GSRC,h)\int_a^b f(t)dt,$$

respectively.

It has been shown [1] that for proper choice of h_1 and h_2,

$$(GRC,h_1)\int_a^b f_n(t)dt = f_{n-1}(b)-f_{n-1}(a)$$

and

$$(GRC,h_2)\int_a^b C_nDf(t)dt = f(b)-f(a),$$

if f_n and C_nDf are finite everywhere in $[a,b]$. Similarly [1] under natural assumptions it is true that the generalized symmetric Riemann complete integral integrates everywhere finite de la Vallee Poussin and SC_n-derivatives.

We obtain further the following result:

Theorem. If f is finite and C_nP-integrable on $[a,b]$ then f is generalized Riemann complete integrable with respect to (a suitable choice of a pair of interval functions) $A = \{\varphi_\ell,\varphi_r\}$ and the integrals are equal.

References

1. Cross, G.E., Higher order Riemann complete integrals, Real Anal. Exchange, 11 (1985-86), 347-364.

2. McGrotty, J., A theorem on complete sets, J. London Math. Soc., 37 (1962), 338-340.

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