CONCERNING EXTENDABLE CONNECTIVITY FUNCTIONS, A CONTINUATION

By Jerry Gibson

This talk will be a continuation of the talk given last year at the Ninth Symposium, [4]. But first we give a brief review.

In the classic paper [14], J. Stallings asked the following question: "If $I = [0,1]$ is embedded in I^2 as $I \times 0$, can a connectivity function $I \to I$ be extended to a connectivity function $I^2 \to I$?" Negative answers were given to this question by Cornette [3] and Roberts [13]. Each constructed a connectivity function $I \to I$ that is not an almost continuous function.

Definition 1. Let $f:X \to Y$ be a function. Then
(1) f is an almost continuous function provided that every open set containing the graph of f contains the graph of a continuous function with the same domain;
(2) f is a connectivity function provided that if C is a connected subset of X, then the graph of f restricted to C is a connected subset of $X \times Y$; and
(3) f is a peripherally continuous function provided that if $x \in X$ and U and V are open subsets of X and Y containing x and $f(x)$, respectively, then there exists an open set W such that $x \in W \subset U$ and $f(bd(W)) \subset V$ where bd = boundary.

If $f:I \to I$ is an almost continuous function, then f is a connectivity function, [14]; and if f is a connectivity function, then f is a peripherally continuous function. However, if $g:I^n \to I$, $n \geq 2$, then connectivity functions and peripherally continuous functions are the same, [10]. But if $g:I^n \to I$, $n \geq 2$, is a connectivity function, then g is an almost continuous function, [14].

Thus it follows that a connectivity function $I \to I$ that is not an almost continuous function can not be extended to a connectivity function $I^2 \to I$. K. R. Kellum has shown in [11] that every almost continuous function $I \to I$ can be extended to an almost continuous function $I^2 \to I$. Thus a natural question arises.

Question 0. Can an almost continuous function $I \to I$ be extended to a connectivity function $I^2 \to I$?

To give a negative answer to this question, we need the following definitions.

Definition 2. Let f be a real-valued function defined on an interval. Then

(4) f is said to have the Cantor intermediate value property(CIVP) provided that if $p \neq q$ and $f(p) \neq f(q)$, then for each Cantor set K between f(p) and f(q) there exists a Cantor set C between p and q such that $f(C) \subset K$;

(5) f is said to have the weak Cantor intermediate value property (WCIVP) provided that if $p \neq q$ and $f(p) \neq f(q)$, then there exists a Cantor set C between p and q such that $f(C)$ is between $f(p)$ and $f(q)$; and

(6) f has a perfect road at the point x provided that there exists a perfect set P such that x is a bilateral point of accumulation of P and $f|P$ is continuous at x. At the endpoints replace the bilateral condition with a unilateral condition.

In a paper [5] that appeared in the 1982 Topology Proceedings, Fred Roush and I defined the CIVP and constructed an almost continuous function $I \to I$ that did not have the CIVP. In another paper [6] that will appear in the 1985 Topology Proceedings, we defined the WCIVP and proved that if $g:I^2 \to I$ is a connectivity function, then the restriction $g|I \times 0$ has the WCIVP. Moreover, $g|I \times 0$ restricted to the Cantor set is continuous. The almost continuous function constructed in the example mentioned above does not have the WCIVP, and hence can not be extended to a connectivity function $I^2 \to I$.

We also proved in a paper [7] that appeared in the 1985-86 Real Analysis Exchange that if $g:I^2 \to I$ is a connectivity function, then the restriction $g|I \times 0$ has a perfect road at each point where I is embedded in I^2 as $I \times 0$. Thus it follows that if $I \to I$ is an almost continuous function that does not have a perfect road at each point, then it can not be extended to a connectivity function $I^2 \to I$.

We should note that we could restrict g to $I \times p$ for any $p \in I$ and have the same results.

Using a category argument, Fred and I have also constructed in a paper [8] to appear in the Real Analysis Exchange a connectivity function $g:I^2 \to I$ such that for some $p \in I$, $g|I \times p$ is not Marczewski measurable; i.e., there exists a perfect set $Q \subset I \times p$ such that for no perfect subset P of Q is it true that $(g|I \times p)|P$ is continuous. The set of p's for which $g|I \times p$ is not Marczewski measurable is a set of the second category.

The following questions were stated in [4].

Question 1. Does there exist an almost continuous function $f:I \to I$ that has a perfect road at each point but can not be extended to a connectivity function $g:I^2 \to I$?

Unknown

Question 2. Does there exist a Baire class 1 connectivity function $f:I \to I$ that can not be extended to a connectivity function $g:I^2 \to I$?

Jack Brown showed in [1] that for Baire class 1 functions $I \to I$, almost continuous functions and connectivity functions are the same. Recently, Jack Brown, Paul Humke, and M. Laczkovich proved that for Baire class 1 functions $I \to I$, extendable connectivity functions and connectivity functions are the same.

Question 3. If $g:I^2 \to I$ is an onto connectivity function and $f:I \to I$ is a function such that $f \circ g:I^2 \to I$ is a connectivity function, is f continuous except perhaps at 0 or 1?

At a real variables conference at Auburn University, Harvey Rosen answered this question in the affirmative. In fact g can be made a Darboux function and the conclusion is still true.

Question 4. If $g:I^2 \to I$ is an extension of $f:I \to I$ and g is a connectivity function, does f have the CIVP?

Fred Roush and I believe that we have answered this question in the affirmative. In fact f restricted to the Cantor set in the definition will be continuous. We have a partial answer at least.

Question 5. Is it true that if $f:I \to I$ can be extended to a connectivity function $g:I^2 \to I$, then f can be extended to a connectivity function $g:I^2 \to I$ such that g is continuous on the complement of $I \times 0$?

This question will be answered in the affirmative by giving a characterization of connectivity functions $I \to I$ that are extendable to connectivity functions $I^2 \to I$. This result is contained in a paper [9] that will be submitted to the Real Analysis Exchange.

Definition 3. Let $f:I \to I$ be a function. A family of peripheral intervals for f consists of a sequence of ordered pairs (I_n, J_n) of subintervals of I such that

(1) I_n is open in I and the length of I_n converges to 0;

(2) for each x in I and for any $\varepsilon > 0$ there exists (I_n, J_n) such that x is in I_n, the lengths of I_n and J_n are less than ε, and J_n is a subset of $(f(x) - \varepsilon, f(x) + \varepsilon)$;

(3) both endpoints of I_n maps into J_n; and

(4) if I_n and I_m have points in common but neither is a subset of the other, then J_n and J_m have points in common.

Theorem 1. If a family of peripheral intervals exists for $f:I \to I$, then f is the restriction of a connectivity function $g:I^2 \to I$ such that g is continuous on the complement of $I \times 0$.

Lemma. Let $g:I^2 \to I$ be a connectivity function (= peripherally continuous). If g(bd(U) is a subset of an ε-nbhd of g(x), then there exists an interval of the form $[i/2^k, (i+2)/2^k]$ containing both g(x) and g(bd(U)) and having length less than 1/n where $\varepsilon = 1/8n$.

Theorem 2. The existence of a family of peripheral intervals is both necessary and sufficient that a function $f:I \to I$ be the restriction of a connectivity function $g:I^2 \to I$ where $I = I \times 0$.

Remark. Most of the work that Fred Roush and I have done concerning extendable connectivity functions occurred while I was a member of a real variables seminar at Auburn University, Auburn, Alabama. This seminar was headed by Jack B. Brown to whom I am indebted for his expertise in the area of non-continuous real functions and for his willingness to listen to our problems.

REFERENCES

1. J. B. Brown, "Almost continuous Darboux functions and Reed's pointwise convergences criteria," Fund. Math. 86(1974), 1-7.
2. ______ and G. V. Cox, "Classical theory of totally imperfect spaces," Real Analysis Exchange 7(1982), 1-39.
3. J. L. Cornette, "Connectivity functions and images on Peano continua," Fund. Math. 58(1966), 183-192.
4. J. Gibson(R. G. Gibson), "Concerning extendable connectivity functions," Real Analysis Exchange 11 no. 1(1985-86), Proc. of the Ninth Symposium, 56-63.
5. R. G. Gibson and Fred Roush, "The Cantor intermediate value property," Topology Proceedings 7(1982), 55-62.
6. ____________, "Concerning the extension of connectivity functions," Topology Proceedings 10(1985).
7. ____________, "Connectivity functions with a perfect road," Real Analysis Exchange 11 no. 1(1985-86), 260-264.
8. ____________, "The restrictions of a connectivity function are nice but not that nice," Real Analysis Exchange, (to appear).
9. ____________, "A characterization of extendable connectivity functions," Real Analysis Exchange, (to be submitted).
10. M. R. Hagan, "Equivalence of connectivity maps and peripherally continuous transformations," Proceedings of A. M. S. 17(1966), 175-177.
11. K. R. Kellum, "The equivalence of absolute almost continuous retracts and ε-absolute retracts," Fund. Math. 96(1977), 229-235.

12. _______ and B. D. Garrett, "Almost continuous real functions," Proceedings of A. M. S. 33(1972), 181-184.

13. J. H. Roberts, "Zero-dimensional sets blocking connectivity functions," Fund. Math. 57(1965), 173-179.

14. J. Stallings, "Fixed point theorem for connectivity maps," Fund. Math. 47(1959), 249-263.

Jerry Gibson

Department of Mathematical Sciences

Columbus College

Columbus, Georgia 31993

U. S. A.