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CLASSICAL PROBLEMS IN ANALYSIS AND NEW INTEGRALS

I. The classical approach

We shall motivate and describe recent attempts to define, through the use of suitable limits of Riemann sums, integrals which can integrate the divergence of mere differentiable vector fields. Their approach via nonstandard analysis will be sketched in Part II.

If $\Omega \subset \mathbf{C}$ is an open domain and $f: \Omega \to \mathbf{C}$ is $\mathbf{C}$-differentiable, the Cauchy theorem asserts that

$$\int_{\partial I} f(z)dz = 0 \qquad (1)$$

for each 2-interval I whose closure is contained in Ω. Two types of proof are generally offered in textbooks. The first type uses the Green formula to transform the real and imaginary part of the left-hand side in (1) into integrals on I whose integrands are equal to zero by the Cauchy-Riemann formulas. Such an approach requires more regularity on f than the $\mathbf{C}$-differentiability, for example f continuously $\mathbf{C}$-differentiable. The second type of proof, which goes back to Goursat (1900) [3] proves (1) under the mere $\mathbf{C}$-differentiability assertion on f by a contradiction argument and the technique of successive divisions of I. Such a proof is similar to that of a lemma stated and proved in $\mathbb{R}^2$ in 1895 by Cousin and which has played an important role since the late fifties in the definitions of Perron-type integrals through Riemann sums introduced by Kurzweil [9] and Henstock [4]. To formulate precisely the lemma in $\mathbb{R}^n$, let $I =]a_1, b_1] \times \ldots \times]a_n, b_n]$ be a right-closed interval in $\mathbb{R}^n$, $\bar{I}$ its closure and let us call <u>gauge</u> on $\bar{I}$ any positive function defined on $\bar{I}$ <u>L-partition</u> Π of I any finite family

$$\Pi = \{(x^1, I^1), \ldots, (x^q, I^q)\}$$

such that the right-closed subintervals I^j of I partition I and the x^j are points of $\overline{I}$ $(1 \leq j \leq q)$. If, moreover, $x^j \in \overline{I}^j$ $(1 \leq j \leq q)$ then the L-partition Π will be called a <u>P-partition</u> of I. Given a gauge δ on $\overline{I}$ and a L-partition Π of I, we shall say that Π is <u>δ-fine</u> if $I^j \subset B[x^j; \delta(x^j)]$ $(1 \leq j \leq q)$, where $B[y; r]$ is the closed ball of center y and radius $r > 0$ for the norm $|x| = \max_{1 \leq i \leq n} |x_i|$, <u>Cousin's lemma</u> may then be stated as follows.

<u>Lemma 1</u>. For each gauge δ on $\overline{I}$ there exists a δ-fine P-partition of I.

In fact, the usual proof by successive divisions of I provides the supplementary useful information that there exists a δ-fine P-partition $\Pi = \{(x^1, I^1), \ldots, (x^q, I^q)\}$ of I such that each I^j is similar to I (see e.g. [10] for the details).

The direct use of this lemma makes the proof of Goursat's theorem quite straightforward. By the $\mathbb{C}$-differentiability of f over Ω, if $\epsilon > 0$ is given, there exists a gauge δ on $\overline{I}$ such that $y \in \overline{I}$, $z \in \Omega$ and $|z - y| \leq \delta(y)$ imply that

$$(2) \qquad |f(z) - f(y) - f'(y)(z - y)| \leq \epsilon |z - y|.$$

If $\{(z^1, I^1), \ldots, (z^q, I^q)\}$ is a δ-fine P-partition of I with the I^j similar to I, then (1) is equivalent to

$$\sum_{j=1}^{q} \int_{\partial I^j} f(z)dz = 0.$$

On the other hand,

$$(3) \int_{\partial I^j} f(z)dz = \int_{\partial I^j} [f(z^j)+f'(z^j)(z-z^j)]dz + \int_{\partial I^j} [f(z)-f(z^j)-f'(z^j)(z-z^j)]dz$$

$$(1 \leq j \leq q)$$

and the first term in the right-hand member of (3) is equal to zero by direct calculation. By (2) and the δ-fineness of Π, we have

$$\left|\int_{\partial I^j} [f(z) - f(z^j) - f'(z^j)(z - z^j)]dz\right| \le \epsilon\int_{\partial I^j} |z - z^j|d\ell$$

and hence Goursat's theorem will be proved if we can show that

$$\sum_{j=1}^{q}\int_{\partial I^j} |z - z^j|d\ell \le C \tag{4}$$

for some constant $C > 0$ independent of Π. Now, all the I^j being similar to I we have, denoting the length of the largest (resp. smallest) side of I^j by L_j (resp. ℓ_j), and similarly for I,

$$\sum_{j=1}^{q}\int_{\partial I^j} |z - z^j|d\ell \le \sum_{j=1}^{q} 2L_j(L_j + \ell_j) = 2\sum_{j=1}^{q} (\frac{L}{\ell} + 1)L_j\ell_j$$
$$= 2(\frac{L}{\ell} + 1)L\ell$$

(we have used the fact that $L\ell_j = L_j\ell$), and the proof is complete.

A careful analysis of this proof suggests that a modification of the Kurzweil-Henstock approach could lead to an integral which will integrate the divergence of merely differentiable vector fields in $\mathbb{R}^n$ in the same way as the Perron integral in $\mathbb{R}$ integrates all derivatives. If $\mathcal{L}(\dot{I})$ denotes the set of all L-partitions of $\dot{I}$ let us introduce a function $\Sigma: \mathcal{L}(I) \to \mathbb{R}_+$ which will measure the <u>irregularity</u> of a L-partition (examples are given later). Let $I \subset \mathbb{R}^n$ be a right-closed interval and $f: \overline{I} \to \mathbb{R}^p$ a function. If $\Pi = \{(x^1, I^1), \ldots, (x^q, I^q)\}$ is a L-partition of I, we denote by

$$S(f, \Pi) = \sum_{j=1}^{q} f(x^j)m_n(I^j)$$

the corresponding <u>Riemann-sum</u>, where $m_n(I^j) = \prod_{i=1}^{n}(b_i^j - a_i^j)$ for $I^j =]a_1^j, b_1^j] \times \ldots \times]a_n^j, b_n^j]$ $(1 \le j \le q)$.

<u>Definition 1</u>. We say that f is Σ-P-integrable over $\overline{I}$ (resp. <u>Σ-L-integrable over</u> $\overline{I}$) if there exists $J \in \mathbb{R}^p$ such that, for each $\epsilon > 0$

and for each $\eta > 0$, there exists a gauge δ on $\bar{I}$ such that

$$|S(f, \Pi) - J| \leq \epsilon$$

provided Π is a δ-fine P-partition (resp. L-partition) of I with $\Sigma(\Pi) \leq \eta$.

Such a J is necessarily unique and is noted $(\Sigma P)\int_{\bar{I}} f$.

The properties of the integrals given by Definition 1 strongly depend upon the choice of Σ and of the allowed partitions. So, for $\Sigma = 1$, the Σ-P-integral is the multi-dimensional Perron integral introduced independently by Kurzweil [9] and Henstock [4] and the Σ-L-integral is McShane's version of the Lebesgue integral [13]. Defining $\sigma(K)$ for $K =]c_1, d_1] \times \ldots \times]c_n, d_n]$ by

$$\sigma(K) = [\max_{1 \leq i \leq n} (d_i - c_i)]/[\min_{1 \leq i \leq n} (d_i - c_i)]$$

and $\Sigma_0(\Pi)$ by

$$\Sigma_0(\Pi) = [\max_{1 \leq j \leq q} \sigma(I^j)]/\sigma(I)$$

we obtain integrals introduced in [11] as the <u>GP</u>- and <u>GL-integrals</u>, respectively. The GL-integral is shown to be equivalent to the Lebesgue integral. For the GP-integral, the following version of the <u>Stokes theorem</u> can be proved by an argument very similar to that used above to get the Goursat theorem [11].

<u>Theorem 1</u> Let f be a function of $\mathbb{R}^n$ into $\mathbb{R}^n$ which is differentiable on an open domain Ω and let

$$\omega_f = \sum_{i=1}^{n} (-1)^{i-1} f_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n.$$

Then $\operatorname{div} f = \sum_{i=1}^{n} D_i f_i$ is GP-integrable over $\bar{I}$ for each right-closed interval $I \subset \mathbb{R}^n$ such that $\bar{I} \subset \Omega$, and, moreover,

$$\int_I \operatorname{div} f = \int_{\partial I} \omega_f$$

where ∂I denotes the usual oriented boundary of I.

Thus the GP-integral, which reduces to the Perron integral for n = 1, integrates the divergence of merely differentiable vector fields in the same way as the Perron integral integrates every derivative. Also, Theorem 1 provides a "Green-type" proof of the Cauchy theorem for complex functions under the mere $\mathbb{C}$-differentiability assumption.

Although the GP-integral has many of the useful properties of usual integrals, the question of the GP-integrability over $\bar{I}^1 \cup \bar{I}^2$ when f is GP-integrable over abutting intervals $\bar{I}^1$ and $\bar{I}^2$ was left open in [11], as well as the obtention of a dominated convergence theorem (a monotone convergence theorem was proved in [11]). Jarnik, Kurzweil and Schwabik [5] gave a counterexample to the first property and, very recently, Shu Shen Fu [2] has proved the dominated convergence theorem for the GP-integral and has given a sufficient condition for the GP-integral to be additive on abutting intervals. In their paper [5] and subsequent ones [6, 7], Jarnik, Kurzweil and Schwabik introduced a new irregularity function, namely

$$\Sigma_1(\Pi) = \sum_{j=1}^{q} \int_{I^j} |x - x^j| \, dm_{n-1}(x) \tag{5}$$

where m_{n-1} is the usual (n-1)-dimensional Lebesgue measure and the integrals are Riemann integrals. One can see that this is just what corresponds to the quantity in the left-hand member of (4). The corresponding Σ_1-P- and Σ_1-L-integrals are the $\underline{M}_1$- and $\underline{M}_2$-integrals of [5]. Both allow a Stokes theorem identical to Theorem 1, have the additivity property and the monotone and dominated convergence theorems hold for them. [5] also contains an example showing that Fubini's theorem fails for the GP-, M_1- and M_2-integrals, and it is proved that the following strict inclusion properties hold for $n > 1$ with L for Lebesgue and KH for Kurzweil-Henstock:

$$GP \supset M_1 \begin{array}{c} \supset \ KH \ \supset \\ \\ \supset \ M_2 \ \supset \end{array} L.$$

The M_2-integral definition motivated Jarnik and Kurzweil [7] in defining a non-absolutely convergent integral which allows a Stokes theorem for merely differentiable $(n-1)$-forms with compact support on a n-manifold of class C^1. They do it by introducing the concept of PU-partition of a compact set $M \subset \mathbb{R}^n$, namely a finite family $\Delta = \{(x^1, \theta^1), \ldots, (x^q, \theta^q)\}$ where $x^j \in M$, $\theta^j: \mathbb{R}^n \to [0, 1]$ are C^1-functions with compact support, $0 \leq \sum_{j=1}^{q} \theta^j(x) \leq 1$, $M \subset \text{int}\{x \in \mathbb{R}^n: \sum_{j=1}^{q} \theta^j(x) = 1\}$ $(1 \leq j \leq q,\ x \in M)$. If δ is a gauge on M, Δ is said to be δ-fine if

$$\text{supp}\, \theta^j \subset B[x^j, \delta(x^j)] \quad (1 \leq j \leq q)$$

and, in analogy with (5), the irregularity function for the PU-partitions is defined by

$$\Sigma_2(\Delta) = \sum_{j=1}^{q} \int_{\mathbb{R}^n} |x - x^j| \sum_{i=1}^{n} |D_i \theta^j| dx.$$

For $f: \mathbb{R}^n \to \mathbb{R}^p$ with compact support, the corresponding Riemann sum is given by

$$S(f, \Delta) = \sum_{j=1}^{q} f(x^j) \int_{\mathbb{R}^n} \theta^j(x) dx.$$

Definition 2. ([7]) We say that f is PU-integrable if there exists $J \in \mathbb{R}$ such that for each $\epsilon > 0$ and each $\eta > 0$, there is a gauge δ on supp f such that

$$|S(f, \Delta) - J| \le \epsilon$$

provided Δ is a δ-fine PU-partition of supp f with $\Sigma_3(\Delta) \le \eta$.

An important additional property of the PU-integral is that the change of variables theorem in its usual form holds for it [7]. In a very recent work [8], Jarnik and Kurzweil have modified the above definition to allow a Stokes theorem when the differentiability property may fail on some sufficiently small subsets of the manifold.

Independent significant contributions to the above-mentioned problems have been made by W. Pfeffer [15, 16] and are described in detail in his contribution to the conference. His approach is closer in spirit to the use of the irregularity function Σ_0 but involves a modification which allows us to overcome the lack of additivity of the GP-integral. Let d(K) denote the diameter of a right-closed interval $K \subset \mathbb{R}^n$ and $r(K) = m_n(K)/(d(K))^n$ its <u>regularity</u>. Notice that $1/\sigma(K) \ge r(K) \ge 1/(\sigma(K))^{n-1}$. If $0 \le k \le n - 1$ and H is a <u>k-plane</u> (a k-dimensional linear submanifold of $\mathbb{R}^n$ parallel to k distinct axes), $r_H(K)$ is defined by r(K) if $H \cap K = \emptyset$ and by $m_k(K \cap H)/(d(K))^k$ if $H \cap K \ne \emptyset$. If $\mathcal{K}$ is a family of planes, let $R_{\mathcal{K}}(K) = \sup_{H \in \mathcal{K}} r_H(K)$ and if $\Pi = \{(x^1, I^1), \ldots, (x^q, I^q)\}$ is a P-partition of I, let

$$\rho(\mathcal{K}, \Pi) = \min_{1 \le j \le q} R_{\mathcal{K}}(I^j).$$

Pfeffer's definition then goes as follows for a function $f: \bar{I} \to \mathbb{R}^p$.

<u>Definition 3</u>. f is <u>ρ-integrable over</u> $\bar{I}$ if there exists $T \in \mathbb{R}^p$ such that for each $\epsilon \in]0, 1/2]$ and each finite family $\mathcal{K}$ of planes there is a gauge δ on $\bar{I}$ such that

$$|S(f, \Pi) - J| \le \epsilon$$

provided Π is a δ-fine P-partition of I with $\rho(\mathcal{K}; \Pi) \ge \epsilon$.

For this integral, Theorem 1 can be proved when f is continuous on $\bar{I}$ and differentiable on int I. See [15] for more details, and [16] for Stokes theorem on differentiable manifolds.

Definitions 1 to 3 show that the basic ingredient in getting generalized Riemann integrals which can integrate merely differentiable vector fields is to introduce in the definition a non-uniformity with respect to some measure of the regularity of the pointed partitions. The conceptual simplicity of the proof of the quite general Stokes theorems obtained in this way is another argument in favor of the Riemann sums approach to integration. Abandoning the uniformity in the size of the intervals in the partitions required in the Riemann integral by using nonconstant gauges had provided integrals of the Lebesgue and Perron power. Abandoning in those integrals the uniformity on the shape of the intervals in the partitions gives multiple nonabsolutely convergent integrals having interesting properties.

II. The nonstandard approach

We shall shortly describe nonstandard characterizations, in the setting of Nelson's internal set theory [14], of the non-absolutely convergent multiple integrals defined in Part I, by following the main lines of [12]. When this paper [12] was in press, we learned from Professor Henstock that generalized Riemann integrals in one dimension had been studied via Nelson's approach by Benninghofen [1] where the reader can find interesting results. In contrast to [1], the presentation given in [12] does not depend upon Benninghofen-Richter theory of superinfinitesimals and is based directly on Nelson's fundamental axioms. In addition, [12] puts emphasis upon the various definitions of Riemann-type for multiple integrals. We refer to [12] for more details and complete proofs and we use the terminology and notations of Part I.

Nelson's internal set theory (IST) starts with the usual Zermelo-Fraenkel set theory with the axiom of choice (ZFC) and adjoin to the usual undefined binary predicate $\in$ a new undefined unary predicate standard (st); we write st(x) for "x is standard", where x is a set. IST uses formulas written with the usual symbols of formal logic and the predicates. A formula of IST is called internal if it does not involve the new predicate "st", and external otherwise. Constants are introduced to simplify the language (like $\emptyset$, $\mathbb{N}$, U, ...) and can always be replaced by formulas using only the formal language. Such a constant is called standard if its definition does not involve "st" and nonstandard otherwise. Thus, all the constants of ZFC are standard. An internal formula is called standard if it contains only standard constants, and nonstandard otherwise. We shall use the following abbreviations:

$(\forall^{st}x)F(x)$ for $(\forall x)(st(x) \Rightarrow F(x))$

$(\forall^{fin}x)F(x)$ for $(\forall x)(x \text{ finite} \Rightarrow F(x))$

and corresponding ones with existential quantifiers. The three following axioms, to be added to those of ZFC, govern the use of the new predicate "st". One can prove [14] that IST is a conservative extension of ZFC.

A. The transfer principle (T)

For any standard formula $A(x, t_1, \ldots, t_n)$ containing no other free variable than $x, t_1, \ldots, t_n$, one has:

$$(\forall^{st}t_1)(\forall^{st}t_2)\ldots(\forall^{st}t_n)[(\forall^{st}x)A(x, t_1,\ldots,t_n) \Leftrightarrow (\forall x)A(x, t_1,\ldots,t_n)]$$

(or equivalently)

$$(\forall^{st}t_1)(\forall^{st}t_2)\ldots(\forall^{st}t_n)[(\exists x)A(x, t_1, \ldots, t_n) \Leftrightarrow (\exists^{st}A(x, t_1, \ldots, t_n]).$$

B. The principle of idealization (I)

For any internal formula $B(x, y)$ with free variables x, y and possibly others, one has

$$(\forall^{st\ fin}z)(\exists x)(\forall y \in z)B(x, y) \Leftrightarrow (\exists x)(\forall^{st}y)B(x, y).$$

C. The principle of standardization (S)

For any formula $C(z)$ with free variable z and possibly others, one has

$$(\forall^{st}x)(\exists^{st}y)(\forall^{st}z)[z \in y \Leftrightarrow (z \in x) \wedge C(z)].$$

Axiom (T) implies that every specific object of conventional mathematics is a standard set and that standard sets are equal if and only if they have the same standard elements. But a standard set may contain nonstandard elements as follows from a consequence of (I): every element of a set v is standard if and only if v is standard and finite. Thus, every infinite set contains a nonstandard element, and this is the case, in particular, for $\mathbb{N}$ and $\mathbb{R}$. Applying (I) to the internal formula (with $\mathbb{R}_+ = \{y \in \mathbb{R}: y > 0\}$ and $|x| = \max_{1 \leq i \leq n} |x_i|$)

$$B(x, y) = (x \in \mathbb{R}^n) \wedge (y \in \mathbb{R}_+) \wedge (|x| > y)$$

we immediately see that the left-hand side is trivially satisfied and hence

$$(\exists x \in \mathbb{R}^n)(\forall^{st} y \in \mathbb{R}_+): \ |x| > y.$$

Such an element is called illimited, and the limited elements are then the $x \in \mathbb{R}^n$ such that

$$(\exists^{st} y \in \mathbb{R}_+): \ |x| \leq y.$$

Now, if x is illimited and n = 1, then 1/x is such that

(6) $$(\forall^{st} y \in \mathbb{R}_+): \ |x| < y$$

(notice that by (T) the inverse of a standard real number is standard) and the elemnts of $\mathbb{R}^n$ satisfying (6) are called infinitesimals. We say that x and y are infinitely close and we write $x \simeq y$ if $x - y$ is infinitesimal.

Recall that a gauge on a set E is a positive application on E, i.e. an element of $\mathbb{R}_+^E$. The following gauges, whose existence will follow from (I), play an important role in our approach. Recall that a gauge $\delta: E \to \mathbb{R}_+$ is standard if its graph is a standard subset of $E \times \mathbb{R}_+$.

Lemma 1. Let $E \neq \phi$ be a standard set. Then there exists a gauge μ on E such that for each standard gauge δ on E and each $x \in E$ one has $\mu(x) \leq \delta(x)$.

To prove this lemma, it suffices to apply (I) to the internal formula $B(\eta, \delta)$ given by

$$(\eta \in \mathbb{R}_+^E) \wedge (\delta \in \mathbb{R}_+^E) \wedge (\eta \le \delta)$$

where $\eta \le \delta$ means $\eta(x) \le \delta(x)$ for all $x \in E$. A gauge μ verifying the conditions of Lemma 1 is called a microgauge on E. It is easy to show that for each $x \in E$, $\mu(x)$ must be infinitesimal but no constant infinitesimal gauge is a microgauge!

Let I be a right-closed standard interval of of $\mathbb{R}^n$ and $\bar{I}$ its closure. A X-partition $\Pi = \{(x^1, I^1), \ldots, (x^q, I^q)\}$ of I as defined in Part I will be called a X-micropartition of I (X = L or P) if Π is μ-fine (as defined in Part I) for some microgauge μ on $\bar{I}$. This implies that q is necessarily unlimited and also that each standard element of $\bar{I}$ is contained in the set $\{x^1, \ldots, x^q\}$ (assume that the standard $c \in \mathbb{R}^n$ is not in this set and use the standard gauge $\delta(x) = |x - c|/2$ and the μ-fineness to get a contradiction). Let $\Sigma: \mathcal{L}(I) \to \mathbb{R}_+$ be an irregularity function defined on the set of all L-partitions of I (see Part I for several examples). A X-micropartition Π will be called Σ-limited if $\Sigma(\Pi)$ is limited (we depart from the terminology of [12] where such a micropartition was called regular.

Theorem 1. Let $f: \bar{I} \to \mathbb{R}^p$ be standard. Then f is Σ-X-integrable over $\bar{I}$ if and only if there exists a (standard) $J \in \mathbb{R}^p$ such that $S(f, \Pi) \simeq J$ whenever Π is a Σ-limited X-micropartition of I.

Proof: *Necessity*. Definition 1, the uniqueness of J and (T) imply that J is standard and such that

$$(7)\quad (\forall\epsilon)(\forall\eta)(\exists\delta)\Big\{(\forall\Pi)[(\epsilon > 0) \wedge (\eta > 0) \wedge (\delta \in \mathbb{R}_+^{\bar{I}}) \wedge (\Pi \in X_\Sigma(\delta, \eta)) \Rightarrow |S(f, \Pi) - J| \le \epsilon]\Big\}$$

where $X_\Sigma(\delta, \eta)$ denotes the set of X-partitions Π of I which are δ-fine and such that $\Sigma(\Pi) \le \eta$. Now (T) applied to (7) with ϵ and η restricted to standard values implies the existence of a standard δ for which $\{\ldots\}$ holds. Now a Σ-limited X-micropartition Π of I will be δ-fine for all standard gauges δ and such that $\Sigma(\Pi) \leq \eta$ for some standard η, so that we have, for such a Π,

$$(\forall^{st}\epsilon > 0): |S(f, \Pi) - J| \leq \epsilon,$$

i.e.

$$S(f, \Pi) \simeq J.$$

Sufficiency. By assumption, taking for δ a microgauge on $\overline{I}$, we have

$$(8) \quad (\forall^{st}\epsilon)(\forall^{st}\eta)\Big\{(\exists\delta)(\forall\Pi)[(\epsilon > 0) \wedge (\delta \in \mathbb{R}_+^{\overline{I}}) \wedge (\Pi \in X_\Sigma(\delta, \eta)) \Rightarrow |S(f, \Pi) - J| \leq \epsilon]\Big\}$$

because the corresponding Π are then Σ-limited X-micropartitions of I. Two consecutive applications of (T) to (8) imply (7) and complete the proof.

Remark 1. By taking $\Sigma \equiv 1$ in Theorem 1, we obtain the nonstandard characterization of the Kurzweil-Henstock (X = P) and Lebesgue integrals (X = L), respectively:

Let $f: \overline{I} \to \mathbb{R}^p$ be standard. Then f is X-integrable over $\overline{I}$ if and only if there exists a (standard) $J \in \mathbb{R}^p$ such that $S(f, \Pi) \simeq J$ whenever Π is a X-micropartition of I.

Remark 2. Call a X-partition Π infinitesimal if $d(I^j) \simeq 0$ for each $1 \leq j \leq q$. Then Robinson [17] has given the following nonstandard characterization of the Riemann integrability of f over $\overline{I}$:

Let $f: \overline{I} \to \mathbb{R}^p$ be standard. Then f is R-integrable over $\overline{I}$ if and only if there exists a (standard) $J \in \mathbb{R}^p$ such that $S(f, \Pi) \simeq J$ whenever Π is an infinitesimal P-partition of I.

As each Σ-limited X-micropartition is an X-micropartition and each X-micropartition an infinitesimal X-partition, the nonstandard characterizations of the above types of integrability have a structure which makes very clear the decrease in generality. Those nonstandard characterizations also make more transparent the proofs of some classical properties (see [12]) although it is not yet clear for the author how much they can simplify the proofs of the convergence theorems.

Let us end this paper by giving a nonstandard characterization of Pfeffer's integral. Let us denote by $\mathbb{H}$ a fixed subset of the set of all finite families of planes and if Π is a P-micropartition of the standard right-closed interval $I \subset \mathbb{R}^n$, let us call Π a <u>ρ-$\mathbb{H}$-appreciable P-micropartition</u> if for each standard $\mathcal{H} \in \mathbb{H}$ $\rho(\mathcal{H}, \Pi)$ is not an infinitesimal (as $\rho(\mathcal{H}, \Pi) \leq 1$, this implies that ρ is neither infinitesimal nor illimited, and such real numbers are called <u>appreciable</u>).

<u>Theorem 2</u>. Let $f: \bar{I} \to \mathbb{R}^p$ be standard. Then f is ρ-integrable over $\bar{I}$ if and only if there exists a (standard) $J \in \mathbb{R}^p$ such that $S(f, \Pi) \simeq J$ whenever Π is a ρ-$\mathbb{H}$-appreciable P-micropartition of I.

<u>Proof</u>: *Necessity*. Again J is necessarily standard and such that

$$(\forall\epsilon)(\forall\mathcal{H})(\exists\delta)\Big\{(\forall\Pi)[(\epsilon \in]0, \tfrac{1}{2}]) \wedge (\mathcal{H} \in \mathbb{H}) \wedge (\delta \in \mathbb{R}_+^{\bar{I}}) \wedge (\Pi \in \mathfrak{R}(\delta,\mathcal{H},\epsilon)) \Rightarrow |S(f, \Pi) - J| \leq \epsilon]\Big\} \tag{9}$$

where $\mathfrak{R}(\delta, \mathcal{H}, \epsilon)$ denotes the set of δ-fine P-partitions such that $\rho(\mathcal{H}, \Pi) \geq \epsilon$. By (T) applied to (9) with ϵ and $\mathcal{H}$ restricted to standard values, we obtain a standard δ for which $\{\ldots\}$ holds. Now if P is a ρ-$\mathbb{H}$-appreciable P-micropartition, if $\mathcal{H} \in \mathbb{H}$ and $\epsilon \in]0, \rho(\mathcal{H}, \Pi)]$ are standard, and if δ is the corresponding standard gauge, Π is δ-fine and

$\rho(\mathcal{X}, \Pi) \geq \epsilon$, so that

(10) $$|S(f, \Pi) - J| \leq \epsilon.$$

Thus (10) holds for all standard $\epsilon \in]0, \rho(\mathcal{X}, \Pi)]$ and hence for all standard $\epsilon > 0$ so that $S(f, P) \simeq J$.

Sufficiency. By assumption

(11) $$(\forall^{st}\epsilon)(\forall^{st}\mathcal{X})\Big\{(\exists\delta)(\forall\Pi)[(\epsilon \in]0, \tfrac{1}{2}]) \wedge (\mathcal{X} \in \mathbb{H}) \wedge (\delta \in \mathbb{R}_{+}^{\bar{I}}) \wedge (\Pi \in \mathfrak{P}(\delta, \mathcal{X}, \epsilon)) \Rightarrow |S(f, \Pi) - J| \leq \epsilon]\Big\}.$$

Indeed, it suffices to take for δ a microgauge on $\bar{I}$ so that Π δ-fine implies that Π is a P-micropartition and $\rho(\mathcal{X}, \Pi) \geq \epsilon$ with $\epsilon > 0$ standard implies that Π is ρ-$\mathbb{H}$-appreciable, so that $S(f, \Pi) \simeq J$ and hence $|S(f, \Pi) - J| \leq \epsilon$. Two successive applications of (T) to (11) then give the result.

Theorems 2 and 3 show that comparing the Σ-X- and ρ-integrability concepts just reduces to comparing the Σ-limited X-micropartitions and the ρ-$\mathbb{H}$-appreciable P-micropartitions. So, the equivalence of the Σ_0-P- and the ρ-Φ-integrals follows immediately from the inequalities between $\sigma(K)$ and $r(I)$.

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