Jan Mařík, Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

CHARACTERISTIC FUNCTIONS THAT ARE PRODUCTS OF DERIVATIVES

Let D be the system of all (finite) derivatives on the real line R. For each set $A \subset R$ let C_A be its characteristic function. Let $\mathfrak{a}$ be the system of all sets $A \subset R$ such that $C_A = fg$ for some $f, g \in D$. (It is not difficult to prove that every closed set belongs to $\mathfrak{a}$.) Since each derivative is a Baire 1 function and since $A = \{x;\ C_A(x) \geqq 1\} = \{x;\ C_A(x) > 0\}$, we see that every set in $\mathfrak{a}$ is ambiguous (i.e. at the same time an F_σ-set and a G_δ-set). Now let $A \subset R$, $f, g \in D$, $C_A = fg$, $p, x_n, y_n \in R$, $p < x_n < y_n$ $(n = 1,2,\ldots)$ and $\liminf \frac{y_n - x_n}{y_n - p} > 0$. Let $f = F'$, $g = G'$. It is easy to prove that $\frac{F(y_n) - F(x_n)}{y_n - x_n} \longrightarrow F'(p)$ $(= f(p))$; similarly for G. Write $J_n = (x_n, y_n)$ and suppose that $J_n \subset A$ for each n. Using the Cauchy inequality and the Darboux property of derivatives we get $(y_n - x_n)^2 = \left(\int_{J_n} \sqrt{fg}\right)^2 \leqq \int_{J_n} f \cdot \int_{J_n} g = (F(y_n) - F(x_n)) \cdot (G(y_n) - G(x_n))$ for each n. Dividing by $(y_n - x_n)^2$ and passing to the limit we obtain $1 \leqq f(p) \cdot g(p) = C_A(p)$ so that $p \in A$. Hence: If $A \in \mathfrak{a}$, $B = R \setminus A$ and $p \in B$, then such intervals J_n do not exist. (Intuitively: There are no essential holes in B close to p.) This (and a "symmetrical" argument) shows that B is nonporous (i.e. nonporous at p for each $p \in B$). Since A is ambiguous if and only if B is, we have the following simple result: If $A \in \mathfrak{a}$, then B is ambiguous and nonporous.

It can be proved that these two properties of B imply that $A \in \mathfrak{a}$. Actually, we have a more precise statement:

Theorem 1. Let $A \subset R$, $B = R \setminus A$. Then the following three conditions 1), 2) and 3) are equivalent to each other:

1) There is a natural number m and functions $f_1, \ldots, f_m \in D$ such that $C_A = f_1 \cdots f_m$.

2) B is ambiguous and nonporous.

3) There are functions $f, g \in D$ such that $f = g = 1$ on A and $fg = 0$ on B.

Let us compare Theorem 1 with an earlier result (see [1], pp. 33–34):

<u>**Theorem 2**</u>. Let $A \subset R$, $B = R \backslash A$. Then the following three conditions 4), 5) and 6) are equivalent to each other:

4) There is a natural number m and nonnegative functions $f_1, \ldots, f_m \in D$ such that $C_A = f_1 \cdots f_m$.

5) B is ambiguous and each point of B is a point of density of B.

6) There are functions $f, g \in D$ such that $f = g = 1$ on A, $0 \leqq f < 2$, $0 \leqq g < 2$ on R and $fg = 0$ on B.

Theorem 2 suggests that it is probably possible to improve or modify Theorem 1 in various ways. (Can we require f to be bounded [nonnegative] in 3)? Can we say more about f and g, if we drop the requirement $f = g = 1$ on A? I was not able to find any reasonable answers to similar questions.)

Reference

[1] Baire one, null functions, A.M. Bruckner, J. Mařík, and C.E. Weil, Contemporary Mathematics, Vol. 42, 1985, 29–41.