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INTEGRATION IN FUNCTION SPACES

R. Henstock's general theory of integration is based on division spaces rather than measure theory (1,2). Division spaces arise as follows. Given a space T and a family of subsets or "intervals" I of T, a partition of T is a finite collection of disjoint intervals I whose union is T. Henstock defines collections S of interval-point pairs (I,x), $x \in T$. A division $\mathcal{E}$ of T from S is a finite subcollection of (I,x) from S such that the intervals I form a partition of T. The conditions satisfied by the collections S include the following.

(i) There exists S containing a division of T. (For such S we say that S divides T.)

(ii) If S_1 and S_2 both divide T then there exists S_3, dividing T, in the intersection of S_1 and S_2.

If f is a real or complex valued function of points x in T and m is, similarly, a function of the intervals I of T, then the integral over T of f with respect to m, which we denote by $\int_T f(x)m(I)$ or $\int_T fdm$, is z where z satisfies the following condition.

Given $\varepsilon > 0$ there exists S dividing T so that, for any division $\mathcal{E}$ of T from S, .

$$\left| z - (\mathcal{E})\sum f(x)m(I) \right| < \varepsilon$$

where $(\mathcal{E})\sum f(x)m(I)$ represents summation over the (I,x) in $\mathcal{E}$ and corresponds to the Riemann sum of Riemann integration. In the latter case, each S is the collection of $([u,v),x)$, u, v real, $x = u$ or v, $v-u < \delta$.

More generally, if $h(I,x)$ is a function of interval-point pairs then the integral z of h over T, denoted by $\int_T h(I,x)$ or $\int_T dh$, exists if z satisfies the above condition with $f(x)m(I)$ replaced by $h(I,x)$.

Given the real interval $(0,t)$ let T be the set of real valued functions x defined on $(0,t)$. Thus T is the product of R by itself uncountably many times. Given a finite subset $N = \{t_1, t_2, \ldots, t_n\}$ of $(0,t)$ and $x \in T$, let $t_0 = 0$, $t_{n+1} = t$, and write $x_j = x(t_j)$, $1 \le j \le n$, $x_0 = 0$, $x_{n+1} = y$ where y is a fixed real number, and let $x(N) = (x_1, x_2, \ldots, x_n)$ so $x(N)$ is a point of R . An interval I of T is the set of x in T satisfying $u_j < x_j < v_j$, $1 \le j \le n$, where u_j and v_j are real numbers.

The division space structure for the function space T is produced as follows. Let A be a countable subset of $(0,t)$. For each x in T let $L(x)$ be a finite subset of A. For each finite subset N of $(0,t)$ containing $L(x)$, let $\delta = \delta(N)$ be positive. Then (I,x) is in S provided $v_j - u_j < \delta$, $1 \le j \le n$, with $x(t_j) = u_j$ or v_j. Thus the elements (I,x) of S are determined by the choice of A, $L(x)$ and (N) for N containing $L(x)$.

Condition (i) above is satisfied as every such S contains divisions of T. The other conditions in the definition of a division space are also satisfied. Thus the integral of any functional h(I,x) in the unrestricted function space is defined. To integrate a functional h over a proper subspace of T such as the space C of continuous functions or paths, we multiply h by the characteristic function of C and integrate the resulting functional over T.

Let $h(x_1, \ldots, x_n)$ be a real or complex valued function of $x_j = x(t_j)$ and let h(I) denote the following function of intervals I of T:

$$h(I) = \int_{u_1}^{v_1} \cdots \int_{u_n}^{v_n} h(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

<u>Theorem 1</u> : If h is integrable in T then $\int_T dh$ is the limit of a sequence of terms

$$\int_{-a_1}^{b_1} \cdots \int_{-a_n}^{b_n} h(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

in which $n, a_1, b_1, \ldots, a_n, b_n$ tend to infinity, taking successively larger positive values.

If c is a complex number, $c = a+ib$, $a \leq 0$, $b \geq 0$, $c \neq 0$, let

$$w(I) = \int_{u_1}^{v_1} \cdots \int_{u_n}^{v_n} \prod_{j=1}^{n+1} \left(\frac{\pi}{-c} (t_j - t_{j-1}) \right)^{-\frac{1}{2}} \exp\left(c \sum_{j=1}^{n+1} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} \right) dx_1 \cdots dx_n .$$

We call w the generalised Wiener integrator.

Theorem 2 : $w(I)$ is integrable in T with

$$\int_T dw = \left(\frac{\pi t}{-c}\right)^{-\frac{1}{2}} \exp\frac{cy^2}{t} .$$

If $c = -1/2$ then the integral of Theorem 2 is the Wiener integral and the function on the right hand side is the diffusion function for a Brownian particle. If $c = i/2$ then we have the Feynman integral of quantum mechanics and the function is the propagator function for a single free particle in one dimension.

References

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