SOME ASPECTS OF DYNAMICAL BEHAVIOR OF MAPS OF AN INTERVAL

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During the past few years a number of books, surveys and expository articles dealing with dynamical behavior of continuous functions have been written by biologists, physicists and mathematicians. The scope of the subject is large, touching on many parts of mathematics: ergodic theory, differential equations, complex analysis, and fractals among others. Those publications having primarily a real analysis flavor stress well-behaved functions. For example, for functions mapping an interval into itself, one usually assumes quite a bit of differentiability of the functions as well as certain other regularity conditions that lead to pleasing developments. Such requirements on the functions are natural since the types of functions that arise in practice meet these conditions. Yet, the functions that serve as models for practical problems in the biological, social and physical sciences may be imperfect, and it seems desirable to study the dynamical behavior of continuous functions that are not so well behaved. In some cases, a slightly imprecise model may cause no serious errors; in others, the errors could be very serious indeed. (Even when the function in use is perturbed to another well-behaved function, the perturbed function may possess dynamical properties very different from those of the unperturbed function.)

This area seems to be one in which the real analyst familiar with the behavior of continuous functions that are not necessarily well-behaved can make contributions.

We shall discuss the dynamics of one dimensional maps, that is, of continuous functions mapping an interval into itself. After a motivational section and an illustrative example, we discuss briefly some of the well-known iterative behavior of functions meeting certain regularity conditions. We then contrast this behavior with the corresponding behavior of the "typical" continuous function. We follow with a discussion of certain aspects of chaotic behavior. We close with various remarks and with suggestions of problems.

The reader interested in the ways the dynamics of one dimensional maps arise in practice in the biological, physical and social sciences may wish to consult the articles [M], [LY] and [VSK] as well as the references cited in the articles. Readable developments of the dynamics of well-behaved functions can be found in the recent books [CE], [D] and [P].

Motivation

To provide an indication of the manner in which a study of the dynamical behavior of a one dimensional map can arise, we begin with a brief discussion of a well-known equation, the logistic equation.

Suppose a populational biologist wishes to study the growth pattern of a particular species. It may be that the main ingredients affecting the growth are the size of the initial population and environmental factors that limit the growth. In that case, one often arrives at the differential equation $y' = ky(L-y)$, where $y = y(t)$ is the size of the population at time t, k is a constant and L is the limiting population. For an initial population $y_0 < L$, one finds $y' > 0$ and finds $y(t) \uparrow L$. Similarly, if $y_0 > L$, one finds $y(t) \downarrow L$. An investigator who makes a slight error in estimating y_0, k, or L may find no serious error in predictions concerning the long-term growth pattern of the population.

This model assumes continuous growth of the population. But some species may develop in discrete generations and not procreate continuously. If the size of a particular generation is near the limiting size L, the size of the next generation may exceed L, only to die quickly, leaving only a small population to give rise to the generation to follow. To study such a situation, one wants a discrete model rather than a continuous model.

Population biologists will often turn to the logistic <u>difference</u> equation $x_{n+1} = kx_n(1-x_n)$ as a model to handle discrete growth [M], [D]. Here x_n represents the size of the nth generation as a fraction of some maximal size "1". This allows for the type of growth which involves exceeding the "limiting" size some of the time. But is there a limiting behavior?

To discuss this and related questions, it is convenient to study the iterative behavior of functions of the form $f(x) = kx(1-x)$. Writing $x = x_0$, we find $x_{n+1} = f^{n+1}(x)$, where $f^1 = f$ and, for $n \geqq 1$, $f^{n+1} = f^1 \circ f^n$. For $0 \leqq k \leqq 4$, f maps $[0,1]$ into itself.

We discuss briefly a few aspects of the dynamical behavior of functions in this class. Excellent, rather detailed, accounts can be found in the books [D] and [P].

Let $f(x) = kx(1-x)$, $0 < k \leqq 4$. It is easy to verify that f has fixed points at $x = 0$ and at $x = \frac{k-1}{k}$, if $k > 1$; that $f'(0) = k$, and that $f'(\frac{k-1}{k}) = 2 - k$ $(k > 1)$. If $k < 1$, then $f(x) < x$ for all $x \neq 0$, thus $\lim_{n\to\infty} f^n(x) = 0$ for all x. (The population becomes extinct!)

If $1 < k < 3$, then $-1 < f'(\frac{k-1}{k}) < 1$. Thus sufficiently small intervals containing $\frac{k-1}{k}$ shrink in size. In fact, if $x \neq 0,1$, then $f^n(x) \to \frac{k-1}{k}$. (This can be "seen" geometrically by sketching the graph of f and employing "graphical analysis" ([D], p 20). The population stablizes.)

For $k > 3$, $|f'(\frac{k-1}{k})| > 1$. Thus, small intervals containing $\frac{k-1}{k}$ are mapped onto larger intervals. Points near $\frac{k-1}{k}$ are "repelled" from $\frac{k-1}{k}$ instead of being attracted to this fixed point. The dynamics becomes increasingly complicated as k increases. For k slightly larger than 3, there will be points whose orbits have period 2, but no periodic points of higher period. For example, for $k = 3.1$, the point $\frac{2.1}{3.1}$ is a repelling fixed point. But the pair .55801..., .76456... map onto each other and almost all x are attracted to this orbit (the cluster set of $\{f^n(x)\}$ equals these two points). Thus, the sizes of future populations bounce back and forth between these two values. For larger values of k, periodic points of large period arise and for certain values of k, rather strange behavior occurs. Instead of discussing this behavior for functions in this class, we refer the reader to [D] or [P]. We prefer to illustrate various sorts of behavior in connection with a function that lends itself to simple arithmetic computations.

The Hat Function

$$\text{Let} \quad f(x) = \begin{cases} 2x, & 0 \leqq x \leqq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \leqq 1 \end{cases}$$

This function has been studies by many authors. It is topologically conjugate to the function $4x(1-x)$ on $[0,1]$. This means that there is a homeomorphism h of $[0,1]$ onto itself such that this function equals $h \circ f \circ h^{-1}$. Functions that are topologically conjugate exhibit identical dynamical behavior. We study f because much of its dynmaical behavior is easy to verify since the iterative behavior of f lends itself to binary arithmetic.

Suppose each $x \in [0,1]$ is represented by a binary expansion.

$x = .x_1x_2x_3\ldots$ $\qquad$ ($x_i = 0$ or 1 for all i)

$$\text{Then}\quad x = \begin{cases} .0x_2x_3\ldots & \text{if } 0 \leqq x \leqq \frac{1}{2} \\ .1x_2x_3\ldots & \text{if } \frac{1}{2} < x \leqq 1 \end{cases}$$

One verifies easily that

$$f(x) = \begin{cases} .x_2x_3\ldots & \text{if } 0 \leqq x \leqq \frac{1}{2} \\ .\overset{*}{x}_2\overset{*}{x}_3\ldots & \text{if } \frac{1}{2} < x \leqq 1. \end{cases}$$

Here $\overset{*}{x}_i = 0$ if $x_i = 1$ and $\overset{*}{x}_i = 1$ if $x = 0$.

Thus, for example

$$f(.01100\overline{1100}) = .1100\overline{1100}$$

$$\text{and}\quad f(.1100\overline{1100}) = .01100\overline{1100}$$

i.e. $\frac{2}{5} = .01100\overline{1100}$ is a point of period 2.

Similarly, one finds that for each n, $\dfrac{2}{2^n+1}$ has periodic n.

The function f also exhibits what is called <u>transitive</u> or <u>nomadic</u> behavior. There are points with dense orbit. (An investigator dealing with this function (or the function $4x(1-x)$ would observe behavior that appears random.)

To see this consider an interval

$$I = (.a_1a_2\ldots a_n\overline{0},\ .a_1a_2\ldots a_n1\overline{0}).$$

Let x be any point in $[0,1]$ whose binary expansion includes the "block"

$0a_1 \ldots a_n 0$, say $x = .x_1 x_2 \ldots x_k 0 a_1 \ldots a_n 0 x_{k+n+3} \ldots$

Then $$f^k(x) = .0a_1 \ldots a_n 0 x_{k+n+3} \ldots$$

or

$$f^k(x) = .1\overset{*}{a} \ldots \overset{*}{a_n} 1 \overset{*}{x}_{k+n+3} \ldots$$

In either case, $f^{k+1}(x) = .a_1 \ldots a_n 0 x_{k+n+3} \ldots$

Thus $f^{k+1}(x) \in I$.

Thus, if the binary expansion of x contains <u>every</u> finite block, x has a dense orbit, the sequence $\{f^n(x)\}$ is dense in [0,1]. Of course, almost every x <u>does</u> have a binary expansion including all finite blocks, so almost every x has a dense orbit. It is easy to verify that the set of points that have dense orbits relative to some function is of type G_δ. Thus, a residual subset of [0,1] having full Lebesgue measure, consists of points with dense orbits.

The homeomorphism h giving rise to the conjugacy of f with the function $4x(1-x)$ is absolutely continuous. Thus this function also has a residual set of full measure containing points with dense orbit. If this function serves as a model for the population biologist, our biologist would almost surely observe "random" changes in population size.

The function f also displays another sort of undesirable iterative behavior——Chaos. We shall consider Chaos a little later. First, we discuss briefly the iterative behavior of well-behaved functions.

<u>Well-behaved functions</u>

The class of functions $kx(1-x)$, $0 \leq k \leq 4$ is not the only class that arises naturally in the sciences when dynamical questions of a discrete nature are under consideration. For example, the families $f(x) = kx\, e^{k(1-x)}$

$(k > 1)$ and $f(x) = \sin(kx)$ $(\frac{\pi}{2} < k \leqq \pi)$ arise.

The members of these and other families posess a number of regularity conditions: they have continuous second derivatives in (0,1); f' vanishes only at the maximum m and $f''(m) < 0$; they are unimodal (m is the only local maximum); and they satisfy a certain technical condition [P].

It turns out that much is understood about the dynamics of functions meeting there conditions [P]. We mention only one aspect of what is known, the manner in which the orbits of most points are attracted to certain sets.

Any function f meeting the regularity conditions we mentioned and satisfying $f(z) > z$ on $(0,m]$ posesses exactly one of the following properties:

1. There is a periodic orbit that attracts each point in some dense open subset of [0,1].
2. There is a Cantor set K that attracts each point in some set residual in [0,1].
3. The function f has "sensitive dependence on initial conditions". (This means that there exists $\varepsilon > 0$ such that for every $x \in [0,1]$ and every neighborhood U of x, there exists $n \geqq 0$ such that $|f^k(U)| \geqq \varepsilon$ for all $k \geqq n$.)

For the functions $f(x) = kx(1-x)$, one finds, for example [P] that

when $k = 3.1$, a dense open set is attracted to the orbit $.55801... \leftrightarrow .76456$;

when $k \simeq 3.569...$ (see [P]), there is a Cantor set which attracts a residual subset of [0,1];

and when $k = 4$, we saw there are points with dense orbit. This implies that there is sensitive dependence to initial conditions. In fact, for any nomadic function it is true that there exists intervals K and L such that for each interval I there exists an n such that

$f^k(I) \supset K$ or $f^k(I) \supset L$ for all $k \geq n$. (This follows readily from the proof of Theorem 11 in [BCR].)

Thus the family of functions of the form $f(x) = kx(1-x)$ displays all three of the possibilities for the class of well-behaved functions we mentioned. The same is true of many reasonable families of functions.

Much more is known about the dynamics of well behaved functions. We merely scratched the surface in this section. Readable and highly interesting accounts can be found in the books [CE], [D] and [P]. In particular, our statement concerning the "three types of possible behavior" represents a highly condensed and simplified version of material developed in [P]. The details of the manner in which attraction occurs, the way in which behavior changes as the parameter k is varied, the role of the orbit of the maximum point and other such topics are fascinating.

But not all continuous functions are well-behaved. What can we say if a function is, for example, nowhere differentiable? We offer a contrast in our next section.

The dynamics of typical continuous functions

Suppose a scientist uses a well-behaved function f as a model to describe an iterative process. Even if the process can be fully described by some function g, the function f is undoubtedly only an approximation to g. After all, it is fashionable nowadays to assert that nature is best described by fractals!, [Man]. The definition of "fractal" may vary from one author to another, but surely a well-behaved function isn't a fractal. If a function is a fractal, it must surely be nowhere differentiable and exhibit various other properties common among typical continuous functions. (As usual, by a "typical" property, we mean a property shared by all functions in some residual subset of $C[0,1]$. A typical continuous function is one that has all the typical properties under consideration.)

If the process our scientist is studying requires a typical function g rather than a well-behaved function f for modeling, how serious are the consequences likely to be, assuming, of course, that $\|g-f\|$ is small?

To discuss this problem, we should first indicate the dynamics of a typical continuous g. Might a typical g possess the desirable attractive behavior (1) possessed by some well-behaved functions? What about the pattern (2)? Or must case (3) apply, along with the possibility of nomadic, or near nomadic behavior?

The reader familiar with the behavior of typical continuous functions will surely suspect the answer to all these questions is "NO". Typical behavior is much more complicated, yet much more regular. Typical properties are the same for all typical functions (by definition of "typical") and it would be surprising if there were <u>three</u> different possibilities. The typical function has many periodic points, but collectively they attract only a first category subset of $[0,1]$; there will be c pairwise disjoint attracting Cantor sets, but each attracts only a nowhere dense set; there will be no sensitive dependence on initial conditions (according to our definition of the term); there will be no points whose orbit is dense in some interval. Can one make <u>any</u> statement about the iterative behavior on a residual set? Yes. Each of the attracting Cantor sets attracts a nowhere dense set—but collectively, they attract a residual set. There is a residual subset $Q \subset [0,1]$ such that to each $x \in Q$ corresponds a Cantor set K such that the cluster set of the orbit of x is K. If $x \in K$, then x has an orbit that is dense in

K. The function maps K onto itself homeomorphically.

Thus, the well-behaved f and the typical g exhibit quite different dynamical behavior. One can perhaps gain some intuition by noting some differences concerning the orbits of intervals. Consider, a well-behaved f having a fixed point x_0 such that $|f'(x_0)| < 1$. Since the function f is continuous, it maps sufficiently small intervals I containing x_0 into themselves. The sequence $\{f^n(I)\}$ contracts to x_0. It follows that for $x \in I$, $f^n(x) \to x_0$. For a typical g, this never happens. There are intervals, necessarily containing fixed points, that map into themselves. But the sequence $\{g^n(I)\}$ will contract to a nondegenerate interval that, of course, maps onto itself. Similar comments apply to certain intervals containing periodic points. For typical g, arbitrarily close to any periodic point x_0 there will be other periodic points of arbitrarily high period, and there will be "intervals of arbitrarily high period" arbitrarily near x_0. More precisely, given a positive integer N and $\delta > 0$, there is an interval $I \subset (x_0 - \delta, x_0 + \delta)$ and an $n > N$ such that $f^n(I) \subset I$ but $f^m(I) \cap I = \emptyset$ for $m = 1,\ldots,n-1$. Such an interval I contains periodic points of period n as well as periodic points of higher period. One can choose such a periodic point of higher period and begin the process again. Keeping track of the possibilities shows that each such interval I gives rise to c Cantor sets as well as to c periodic points. (Each periodic point will attract an uncountable first category set, but the totality of points attracted to periodic points is only of the first category. This set has cardinality c in every interval, however.)

Some precise formulations of the typical dynamics can be found in [AB]. Many questions remain open, however.

With this background, we return to our question concerning the scientist with a well-behaved f as a model instead of a typical g. How serious is the error caused by the assumptions that f is well-behaved likely to be? (We should mention here that <u>any</u> error may be serious, even without assuming that the true model involves a function of the form $kx(1-x)$, a slight inaccuracy in selecting the parameter k could lead to vastly different behavior of the iterates. We are more concerned at the moment with the difference of observed behavior due to f being well-behaved while g is typical.)

We can pose the problem in two ways:

a) The scientist has f. If the true g is near f, $\|f-g\|$ small, will the observed behavior obtained by g approximate the behavior predicted by f?

and b) If the scientist is able to obtain f near g, will the behavior predicted by f approximate the true behavior?

These two questions are technically different. In a), we are approximating f by g; in b) we are approximating g by f. In both cases, we assume $\|f-g\|$ small.

An oversimplified and vague answer to both questions is that it may well happen that these differences in behavior between f and g that are due only to the fact that f is well-behaved while g is typical, are microscopic in nature and invisible to the eye or to the computer.

For example, regarding question a), suppose f has a fixed point at x_0 and $|f'(x_0)| < 1$. For sufficiently small closed intervals I containing x_0, we will have $f(I) \subset \mathring{I}$ and a de se open set G will be attracted to x_0. Suppose $x \in G$ and $f^n(x) \in I$ or $n \geq N$. What will our scientist observe if $\|g-f\|$ small? If $\|g-f\|$ i sufficiently small, $g^n(x)$ will also be in I and $g(I)$ will be contained in $\mathring{I}$. It may happen that $g^k(x) \to x_0$, but more likely (in the category sense), the sequence $\{g^k(x)\}$ will move around a Cantor set K that is the cluster set of this sequence. Of course, $K \subset I$, so if I is sufficiently small, the microscopic behavior of $\{g^k(x)\}$ for large k is invisible to the scientist. For practical purposes, $g^k(x) \to x_0$, as suspected. On the other hand, once f is chosen, $\|g-f\|$, while small, doesn't change. There will be points $x \in G$ whose orbits (under f) don't enter I for many, many iterations. If this happens first for $n = n_0$, it may well happen that $g^{n_0}(x) \notin I$. Perhaps $g^{n_0}(x)$ is in an interval J on which the behavior of g is quite different from its behavior in I. The scientist will be confused. This type of error was not due, however to g being typical——it was due to $\|f-g\|$ not being sufficiently small. The fact that g is typical led to errors of a microscopic nature.

The situation may be more interesting when f is nomadic, say $f(x) = 4x(1-x)$. Suppose x has a dense orbit under f and the investigator knows this and expects "random" behavior. A typical g has no points with dense orbit. Most likely the sequence $g^k(x)$ will appear to move randomly for a while, but then enter its microscopic attraction to a Cantor set K. (The observed motion may resemble that of a fly moving through space randomly until it is suddenly caught in a spider web.) The set K might be contained in a very small interval, in which case it will appear to the scientitst that the orbit approaches a fixed point. If the smallest interval containing K can be "seen", then the scientist will observe periodic motion, the magnitude of the observed period depending on the ability of the scientist to distinguish the small intervals that move periodically near K. It may, of course, happen that the number of iterates needed for the sequence $\{g^k(x)\}$ to be trapped near K is so large that it doesn't occur during the observation period. In that case, the scientist may be very pleased in the belief that the random behavior predicted by f has occured.

Regarding question b), we mention that some relevant information is contained in some of the lemmas found in [AB]. The typical g has neighborhoods all of whose members possess the nonmicroscopic behavior of g. (This usually happens when one obtains a residual class of functions via the Baire Category theorem). Thus, once again, the choice of a well-behaved f instead of a typical g may well lead to only invisible errors. (More important errors may well be due to the inability to obtain f sufficiently close to g or to round-off errors inherent in any computer.)

We close this section by mentioning that an interesting result related to question a) for functions f that are well-behaved according to a different criterion has been obtained in [SS]. Roughly speaking, the result states that any continuous g that is sufficiently near a function having periodic points of only finitely many periods and whose set of periodic points is nowhere dense, must have all its orbits asymptotically periodic up to microscopic behavior.

Chaos

We have seen that a finite set might attract each x in some residual set. This happens when the orbit of x is asymptotically periodic. Similarly an uncountable set may serve as an attractor. This set could be a Cantor set or it could be an interval. We have used the family $kx(1-x)$ to illustrate each of these types of attraction. When the Cantor set K attracts a residual set of points, the behavior of the sequence $\{f^n(x)\}$ may eventually appear to be periodic to an observer. The magnitude of the apparent period depends on the observer's ability to distinguish microscopic behavior. When an interval [a,b] is the attractor, for example, when f is nomadic, the behavior may appear "random". It is natural to ask whether it is possible for a countable set A to serve as attractor. If so, how can an orbit be attracted to A?

Let us return for a moment to the hat function. It is clear that for each positive integer k, $f(2^{-k}) = 2^{-k+1}$. Consider now any $x \in (0,1)$ whose binary expansion alternates blocks of 0's and 1's, these blocks becoming increasingly longer. Recalling our simple method of following the orbit of a point under f, we see that for such an x the sequence $\{f^n(x)\}$ is attracted to the countable set $\{0\} \cup \{2^{-k}\}$, $k = 0,1,2,\ldots$. An observer unable to distinguish microscopic behavior will eventually observe the sequence "stopping" at 0, only to begin moving near the sequence $\{2^{-k}\}$ (in the positive direction) after a long pause at 0.

Suppose now that two points $x \neq y$ both have binary expansions of the type mentioned. Suppose also that, infinitely often, x has a long block of 1's in the positions that y has a long block of 0's. One finds that for infinitely many values of n, $|f^n(y) - f^n(x)|$ is near 1. But if there are also infinitely many long blocks that both x and y fill with 0's, then there will be infinitely many values of n for which $|f^n(x) - f^n(y)|$ is near 0. Thus, the orbits of x and y are frequently far apart and frequently close together. One can imagine the confusion this sort of behavior can cause a population biologist. If the initial population is x, which the biologist slightly misjudges to be y, the observed population of future generations will sometimes be very close to the predicted populations and sometimes be very far apart!

The points x and y are part of what is called a "scrambled set".

Definition: S is called a scrambled set for f if whenever $x,y \in S$ $(x \neq y)$,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$$

and $$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

Now, if a scrambled set contains only a few points, it will be unlikely that the true initial population x and the observed initial population y are in S. This suggests the question, "How large can a scrambled set be?"

Li and Yorke [LY] have shown that if there is a point of period 3, there will be an uncountable scrambled set. According to some definitions, this represents "Chaos". (The definition of Chaos varies among authors.) Janková and Smítal [JS] have found other conditions equivalent to chaos. (One of these involves the existence of a certain type of infinite attractor.) They also mention that the existence of a two-point scrambled set implies the existence of an uncountable scrambled set! Thus, the hat function has an uncountable scrambled set. How can one find such a set?

One approach would be to use the same idea we used to obtain the points x and y with long blocks of 0's and 1's in their binary expansions. The difficulty is that each pair chosen from such a set must have the long blocks of 0's and 1's matching up properly. This can, however, be done [BH_1]. One obtains, in fact, a Borel set S having cardinality c in every interval.

Each point of S is attracted to the set $\{0\} \cup \{2^{-k}\}$, $k = 0,1,\ldots$

While the set S is large in cardinality, it is small in measure and in category. With this model, or with the model $4x(1-x)$, the scientist is unlikely to run into chaos because of S. Does the hat function possess a scrambled set of positive measure? The answer is "no". Smital [S_1] has shown that it does possess scrambled sets of full outer measure, but no measurable scrambled set of positive measure. Nor can a scrambled set for f be residual. In fact, any scrambled set for any function must be first category if it has the property of Baire [BH_1]. (There is, however, a scrambled set for f that is second category in every interval [J]. It can't have the property of Baire, of course. More about that later.)

In order for a scrambled set S to be likely to cause confusion for an observer, it would have to have positive measure. This would attach a positive probability to the event that both x (the initial population) and y (the observed initial population), are in S. Can this happen for a function f? Kan [K] and Smital [S_2] simultaneously (but independently) gave examples of this. Kan's example had the additional feature that S is "extremally scrambled":

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1 \quad \text{for all} \quad x,y \in S, \quad (x \neq y).$$

With these models, an investigator would have a positive probability of experiencing chaos. Are there other models for which experiencing chaos is (almost) a certainty, i.e. for which S has measure 1?

Let us return once again to our hat function. The scrambed set S we described has measure 0, but it is a Borel set having cardinality c in every interval. This suffices to guarantee the existence of a homeomorphism h of $[0,1]$ onto itself such that $T = h(S)$ has full measure [G]. The function $g = h \circ f \circ h^{-1}$ has T as an extremally scrambled set.

One can choose h arbitrarily close to the identity and such that each point in the set $\{0\} \cup \{2^{-k}\}$, $k = 0,1,2,\ldots$ remains fixed. The function g is then almost indistinguishable from the hat function. Each point in T is attracted to the set $\{0\} \cup \{2^{-k}\}$, $k = 0,1,2,\ldots$.

The function g illustrates some interesting behavior on T. The sets $g^k(T)$, $k = 0, \pm 1, \pm 2, \ldots$ are pairwise disjoint. Since $T = g^\circ(T)$ has measure 1, all of the other sets $g^k(T)$, $k \neq 0$, have measure 0. The set $T^* = \bigcup_{k=-\infty}^{\infty} g^k(T)$ is also a scrambled set for g and $g^k(T^*) = T^*$ for all k.

One can also show the following, that indicates some of the mixing properties exhibited by the dynamics of g. Let U be any open set whose closure is disjoint from $\{2^{-k}\}$, $k = 0,1,2,\ldots$

Since each point of T is attracted to $\{0\} \cup \{2^{-k}\}$, $k = 0,1,2,\ldots$, no point of $T \cap U$ has an orbit that returns to U infinitely often. Yet, there is a residual subset R of $[0,1]$ each member of which has a dense orbit under g. Thus each point of $R \cap U$ returns to U infinitely often. Thus, U contains two sets, one large in measure, the other in category; the members of the first eventually leave U, never to return, the members of the other return to U infinitely often.

Various definitions of chaos appear in the literature, the notion of scrambled set appearing in one way or another. But the notion of "dense orbit" also carries a sense of chaos. One might ask whether "chaos" and "nomadic behavior" are related. They are. Here is a sample result [BH_1]. If there is a set E, dense in $[0,1]$, such that for $x,y \in E$, $x \neq y$, we have

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1,$$

then f^2 is nomadic. (This implies, of course, that f is nomadic). In the other direction, if f^2 is nomadic then there is a set S, of second category in every interval such that for $x,y \in S$, $x \neq y$, the set

$\{f^n(x) - f^n(y)\}$ is dense in [-1,1]. (This assumes the Continuum Hypothesis.) This, of course implies that S is extremally scrambled. The preceeding actually relates "extremal" chaos to nomadicity of f^2. Changing the notion of chaos or of nomadicity may lead to different sorts of results. If one requires that f (but not f^2) be nomadic, then one cannot conclude extremal chaos, but one can conclude chaos. But chaos does not imply existence of any orbits that are dense in some interval. For example, the typical f is chaotic [BP], but no point has an orbit dense in some interval. (The set of points whose orbits are dense in I is of type G_δ, hence residual in I. But, for typical f, a residual set consists of points attracted to Cantor sets.)

Additional Remarks

We mention briefly a few other items that may be of interest.

1. Equicontinuity of the iterates.

The scientist dealing with an iterative process may have a wish list that looks something like this:

a) The process approaches a state of equilibrium: i.e. $\{f^n(x)\}$ converges to x_0 for all x. In this case, an error in initial measurement has no effect on long term behavior.

b) If a) fails, the process should at least reach some weaker form of equilibrium: $\{f^n(x)\}$ is asymptotically periodic, the same periodic orbit attracting all orbits. If the attracting period is not too long, the long term behavior of the process can still be reasonably understood.

c) At the very least, there should be predictability of long-term behavior: a slight error in initial measurement should not lead to serious errors in predicting long-term behavior. In precise language, the scientist wishes that the family $\{f^n\}$ is equicontinuous.

To what extent are the items on the wish-list likely to occur? Regarding a) and b), it will sometimes happen for well-behaved f that one of these conditions is met for all x in some residual subset of $[0,1]$. (Actually, this set will often have full measure [P]). This may well suffice for practical purposes since the true model g, if near f, may exhibit this behavior, except microscopically. But the conditions will rarely be met for all x.

Regarding c), it is usually too much to expect equicontinuity of the iterates. This rarely happens. If f maps $[0,1]$ onto itself, it will happen if and only if f^2 is the identity: $f^2(x) = x$ for all x. If f is not an onto mapping, there are possibilities, but equicontinuity does impose serious restrictions on f [B], [BH_2]. These conditions may rarely be met in practice. It is, of course, reasonably likely that equicontinuity of the iterates does occur on some large set.

The situation may be summarized by saying that while our scientist's wish list will undoubtedly not be fulfilled, there are reasonable chances that it will be approximated.

2. On periodic behavior.

We have already mentioned that if f has a point of period 3, then f will exhibit chaotic behavior [LY]. It is also true that f will in that case have points of all other periods. This is a special case of a theorem of

Sarkovskii, (see [D] for a development). This theorem provides an ordering of the natural numbers. If k precedes l in this ordering, then every f having a point of period k also has a point of period l. At the head of the list is the number 3. At the end is the number 1. Immediately preceding the number 1 is the sequence $\{2^n\}$ in reverse order. Thus, if there are periods of order 2^n, there will be periods of order 2^m for all $m < n$. There are also examples that illustrate that for each n, there are functions having points of period n but having no points of periods that precede n in the Sarkoviskii order.

Suppose now that we consider a family of well-behaved functions, say $kx(1-x)$, $0 \leqq k \leqq 4$. We have already noted that as k increases, the dynamic behavior becomes more complex. How this occurs is complicated, but we mention one aspect of it related to part of the Sarkovskii order. As k increases, the family experiences "period doubling". This means that there is an increasing sequence of numbers $k_1, k_2, \ldots$ such that as k increases through k_n, the family loses a stable periodic orbit of period 2^{n-1} and gains a stable orbit of period 2^n. For $k_\infty = \lim_{n\to\infty} k_n$ one finds $f(x) = k_\infty x(1-x)$ has an invariant Cantor set K attracting most points of [0,1]. The constant k_∞ is approximately 3.569.

Similar patterns exist for other families of functions depending on a parameter k. A remarkable theorem of Feigenbaum indicates that this occurs in a universal way for certain types of well-behaved families. There is a constant $\delta \simeq 4.6692\ldots$ such that $k_\infty - k_n \sim C\delta^{-n}$, the constant C depending on the family.

The reader interested in Feigenbaum universality may wish to consult [CE] or [VSK]. A readable development of the period-doubling phenomenon can be found in [D], but the Feigenbaum universality is not developed there.

3. Some Problems.

We close with a few problems directly related to material we have discussed. It is possible that these problems are not difficult to solve.

a) One of the three dynamical patterns possible for a well-behaved function is that a Cantor set attracts the orbits of all x in some residual set R. This set R actually has full Lebesgue measure [P]. For a typical continuous f we saw that there exists a family of Cantor sets which, collectively, attract the orbits of all x in some residual set. We have not determined whether this set has full Lebesgue measure.

b) The existence of a large scrambled set creates possible problems for the scientist, as we have observed. For the function g conjugate to the hat function f, the scrambled set T has full measure——someone using such a model will almost surely run into chaos and the resulting impossibility of prediction. Here the term "impossibility of prediction" is being used in a very specific sense: if $x,y \in T$, $x \neq y$, then the orbits of x and y are sometimes close together and sometimes far apart. Actually, most of the time they are close together——both are near zero. Recall, an observer may experience the phenomena $f^{n+1}(x) \simeq 2f^n(x)$ until $f^n(x)$ is near 1: then $f^n(x) \simeq 0$. As the long blocks of 0's in the binary expansion for x become even longer, the orbit of x will appear to "rest" at 0 for longer periods of time. The same is true of the orbit of y. Thus, when predicting $f^n(x)$ for n large, one might simply predict "$f^n(x) = 0$"! In fact $\lim_{n\to\infty} \frac{f(x)+f^2(x)+\ldots+f^n(x)}{n} = 0$ for all $x \in T$. Not only is the limit of averages equal to 0, the limit of variances is also. This raises the question of the limit of correlations. We have not even determined that this limit isn't 1.

More generally, even when extremal chaos is present, it is conceivable that some sort of reasonable prediction is possible. This vaguely-stated question could perhaps be answered in specific cases, or preferably, in general.

c) The countable attractor, $\{0\} \cup \{2^{-k}\}$, $k = 0,1,\ldots,$ present in the dynamics of the hat function, is a closed set with only first order limit points. Which countable closed sets can serve as attractors? Are there examples of all order-types? How do such attractors arise for the typical g? Is there a function that has countable attractors of all order types possible? If so, is this a typical property?

These questions might be easy to answer. We don't know; we haven't given them serious thought.

End of story.

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