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DARBOUX TRANSFORMATIONS

§0. Introduction and basic definitions and notation. Problems connected with properties of Darboux functions were investigated in many papers. An important fact about the class of Darboux functions is that it contains many subclasses of functions, for example, the class of derivatives ([6]) as well as the class of approximately continuous functions ([7]).

In many papers the notion of Darboux function has been generalized to transformations whose domain (and range) are topological spaces more general than the real line (see for example [1], [4], [8], [9], [25]). A detailed account of many generalizations of the notion of Darboux function can be found in article [17].

It is reasonable to require any generalization of the notion of Darboux function to fulfill the following conditions:

I. The fundamental theorems for real Darboux functions of one variable are also true for Darboux transformations in more general spaces.

II. It is possible to consider new problems, which cannot be considered in the case of real functions of a real variable.

III. If the domain of a real Darboux function is the Euclidean space $\mathbb{R}^n$ and $L \subset \mathbb{R}^n$ is an arbitrary line, then $f_{|L} : L \to \mathbb{R}$ is a Darboux function.

The various notions of a Darboux function all involve the image of certain "connected" sets being connected. In this article we study the following specific notion:

We say that $f : X \to Y$, where X and Y are arbitrary topological spaces, is a Darboux transformation, if $f(Ł)$ is a connected set, for every arc $Ł \subset X$.

It is easy to see that the above definition is "similar" to the definitions which are contained in [24], [25], [36].

First we consider problems connected with the results of Z. Zahorski [37] and T. Mańk, T. Swiatkowski [22]. Of course these results are connected with condition I. It is interesting to remark that our notion of Darboux transformation fulfills conditions II. and III. For example we may consider the properties of Darboux transformations $f : \mathbb{R}^2 \to \mathbb{R}^2$ of bounded variation (§2). In particular one can prove that in the space of all bounded Darboux functions $f : I^2 \to \mathbb{R}^2$ of bounded variation, the set of all discontinuous functions is dense and has cardinality 2^c. (Observe that if $f : I \to \mathbb{R}$ is a discontinuous Darboux function, then the variation of f is equal to ∞.) We may also consider the problems connected with the extension of Darboux functions (§3). It is also possible to generalize the notion of a Darboux point (§4) in such a way that one can prove a local characterization of Darboux transformations in arbitrary topological spaces. Moreover we can consider different notions of a Darboux point [2]. One can also consider some interesting applications of these notions.

We shall use the standard notions and notations. By $\mathbb{R}$ we denote the set of real numbers with the natural topology and by I. - an arbitrary closed nondegenerate segment. Let f be an arbitrary transformation. Denote by $C_f(D_f)$ the set of all continuity (discontinuity) points of f. The uniform convergence of a sequence of functions $\{f_n\}$ to f is denoted by $f_n \rightrightarrows f$. The symbol $m_1(m_2)$ denotes 1-dimensional (2-dimensional) Lebesgue measure.

The set of all continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ we denote by $C(\mathbb{R}^2)$. For a function $F = (f_1,f_2) : \mathbb{R} \to \mathbb{R}^2$ we write

$$|F'| = \sqrt{(f_1')^2 + (f_2')^2}.$$

If $f : X \to \mathbb{R}$, then we let $E^a(f) = f^{-1}((-\infty,a))$ and $E_a(f) = f^{-1}((a,+\infty))$ for $a \in \mathbb{R}$. By C_1 we understand the class of all functions $h : \mathbb{R} \to \mathbb{R}$ which have a continuous derivative. If $A \subset \mathbb{R}^2$, then the symbol $\mathrm{proj}_X(A)$ means the projection of the set A on the X-axis. The symbol dia A(card A, $\bar{A}$) denotes the diameter of the set A (cardinality of A, closure of A). By $\rho(a,b)$ we understand the distance between points a and b in the plane. A subset $Ł \subset X$, where X is an arbitrary topological space, is called an arc if there exists a homeomorphism $h : [0,1] \xrightarrow{\text{onto}} Ł$. The elements $h(0)$

and $h(1)$ we shall call the endpoints of $Ł$. The arc with the endpoints x and y we denote by $L(x,y)$. If $Ł$ is an arc and $a,b \in Ł$, then the symbol $L_Ł(a,b)$ denotes the arc with the endpoints at a and b, which is contained in $Ł$.

Let X be an arbitrary topological space. We say that a nonempty, closed set K cuts X into sets U and V (between nonempty sets A and B) if $X \setminus K = U \cup V$, where U and V are nonempty open sets such that $U \cap V = \emptyset$ (and $A \subset U$, $B \subset V$). We say that the nonempty set K quasi-cuts a set $M \subset X$ into sets U and V, between nonempty sets A and B, if $M \setminus K = U \cup V$, where U,V are nonempty separated sets such that $A \subset U$ and $B \subset V$.

Let (Y,d) be a metric space, $A \subset X$ and $f : A \to Y$. Then we say that a transformation $f^* : X \to Y$ is an ε-extension of f (over X) if f^* is an extension of f (i.e. $f^*|_A = f$) and for each $y \in f^*(X)$ there exists $y_0 \in f(A)$ such that $d(y,y_0) < \varepsilon$.

§1. On Zahorski classes of functions of two variables.

In paper [7] A. Denjoy defined approximately continuous functions $f : \mathbb{R} \to \mathbb{R}$. He proved that if f is an approximately continuous function, then it is a Darboux function in Baire class one. In many papers the notion of approximately continuous function has been generalized to a function $f : \mathbb{R}^2 \to \mathbb{R}$. (See [18], [28], [30].) In this article we assume that the base for the definition of density points is the intervals. The set of all approximately continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ is denoted by $\mathfrak{A}^2$. The results of this paper (except Theorem 1.10) are true also if the base for the definition of density points is the cubes. Moreover we give another generalization of the notion of approximately continuous function on $\mathbb{R}^2$ using the arcs as the base for the definition of density points. It is easy to see that our generalization fulfills the conditions I., II. and III. (if we put "approximately continuous functions" in place of "Darboux functions").

Now we pass to the definitions of three classes of arcs in the plane. By $\mathcal{L}_1$ we mean the class of all arcs $Ł \subset \mathbb{R}^2$ such that $m_2(Ł') > 0$ for every arc $Ł' \subset L$. For any $Ł \in \mathcal{L}_1$ let:

$$\delta(Ł) = \sup\{\frac{m_2(Ł)}{m_2(K)} : K \text{ is an interval such that } Ł \subset K\}.$$

For a set $A \subset Ł \in \mathcal{L}_1$ which is measurable relative to 2-dimensional Lebesgue measure we put $m_Ł(A) = m_2(A)$. Let $\mathcal{L}_2$ denote the class of arcs $Ł \subset \mathbb{R}^2$ such that either $Ł$ is a segment parallel to the Y-axis (the family of all such segments we shall denote by I^*) or $Ł \subset \{(p,\varphi(p)) : a' < p < b' \wedge \varphi \in C_1\}$ for some $a',b' \in \mathbb{R}$.

A set $A \subset Ł \in I^*$ is called $Ł$-measurable if A is a measurable set relative to 1-dimensional Lebesgue measure (on $Ł$). A set $A \subset Ł \in \mathcal{L}_2 \setminus I^*$ is called $Ł$-measurable if $\mathrm{proj}_X(A)$ is a Lebesgue measurable set. We say that the set A is $\mathcal{L}$-measurable if A is measurable relative to 2-dimensional Lebesgue measure and $Ł \cap A$ is an $Ł$-measurable set for each $Ł \in \mathcal{L}_2$.

Now we define the function $\Phi : \mathbb{R} \to \mathbb{R}^2$ in the following way:

$$\Phi(p) = (p,\varphi(p)),$$

where $\varphi \in C_1$. Note that if $Ł \in \mathcal{L}_2 \setminus I^*$, then $Ł \subset \{(p,\varphi(p)) : a' < p < b'\}$ and $\varphi \in C_1$ which means that we may define the measure on $Ł$ by

$$m_Ł(A) = \int_{\mathrm{proj}_X(A)} |\Phi'|\, dm_1$$

for every set $A \subset Ł$ such that $\mathrm{proj}_X(A)$ is a measurable set.

If $Ł \in I^*$, then $m_Ł(A)$ denotes 1-dimensional Lebesgue measure. Denote by $\mathcal{L}_3$ the class of all arcs $Ł \subset \mathbb{R}^2$ such that $Ł \notin \mathcal{L}_1 \cup \mathcal{L}_2$. For every $A \subset \mathbb{R}^2$ let

$$m_0(A) = \begin{cases} 0 & \text{if card } A \leqslant \aleph_0, \\ +\infty & \text{if card } A > \aleph_0. \end{cases}$$

For completeness let $m_Ł(A) = m_0(A)$ for every set $A \subset Ł \in \mathcal{L}_3$.

Definition 1.1: Let $A \subset \mathbb{R}^2$ be a Lebesgue measurable set. We say that $p \in \mathbb{R}^2$ is a $\mathcal{L}_1$-density point of A if

$$\lim_{\substack{\mathrm{dia}(Ł) \to 0 \\ \delta(Ł) \to 1 \\ p \in Ł \in \mathcal{L}_1}} \frac{m_Ł(A \cap Ł)}{m_Ł(Ł)} = 1.$$

Definition 1.2: Let $A \subset \mathbb{R}^2$ be an $\mathcal{L}$-measurable set. We say that $p \in \mathbb{R}^2$ is an $\mathcal{L}_2$-density point of A relative to the set $B \subset \mathbb{R}^2$ if for every arc $Ł^* \in \mathcal{L}_2$ such that $p \in Ł^* \subset B$

$$\lim_{\substack{\mathrm{dia}(Ł) \to 0 \\ p \in Ł \subset Ł^*}} \frac{m_{Ł}(A \cap Ł)}{m_{Ł}(Ł)} = 1.$$

In case $B = \mathbb{R}^2$ we say that p is an $\mathcal{L}_2$-density point of A.

Definition 1.3: Let $A \subset \mathbb{R}^2$ be an $\mathcal{L}$-measurable set. We say that $p \in \mathbb{R}^2$ is an $\mathcal{L}$-density point of A if it is an $\mathcal{L}_i$-density point of A (for $i = 1,2$) and moreover $m_0(A \cap Ł) = \infty$ for every arc $Ł$ such that $p \in Ł$.

Definition 1.4: (a) We say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{L}_i$-approximately continuous (for $i = 1,2$) if for every $\alpha \in \mathbb{R}$, $E^\alpha(f)$ and $E_\alpha(f)$ are $\mathcal{L}$-measurable sets such that for each $x_0 \in E^\alpha(f)$ (or $E_\alpha(f)$) x_0 is an $\mathcal{L}_i$-density point of $E^\alpha(f)$ (or $E_\alpha(f)$ respectively). The class of all $\mathcal{L}_i$-approximately continuous functions we shall denote by $\mathcal{L}_i\,\mathcal{C}^2$ (for $i = 1,2$).

(b) We say that a function of $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{L}$-approximately continuous if for every $\alpha \in \mathbb{R}$ $E^\alpha(f)$ and $E_\alpha(f)$ are $\mathcal{L}$-measurable sets such that for each $x_0 \in E^\alpha(f)$ (or $E_\alpha(f)$) x_0 is an $\mathcal{L}$-density point of $E^\alpha(f)$ (or $E_\alpha(f)$ respectively). The class of all $\mathcal{L}$-approximately continuous functions we shall denote by $\mathcal{L}\,\mathcal{C}^2$.

(c) Assume that $Ł \in \mathcal{L}_2$ and $f : Ł \to \mathbb{R}$. We say that a function f is $\mathcal{L}$-approximately continuous if for every $\alpha \in \mathbb{R}$, $E^\alpha(f)$ and $E_\alpha(f)$ are $\mathcal{L}$-measurable sets such that for each $x_0 \in E^\alpha(f)$ (or $E_\alpha(f)$) x_0 is an $\mathcal{L}_2$-density point of $E^\alpha(f)$ (or $E_\alpha(f)$ respectively) relative to $Ł$. The class of all functions $f : Ł \to \mathbb{R}$ which are $\mathcal{L}$-approximately continuous we shall denote by $\mathcal{L}\,\mathcal{C}^{Ł}$.

One can prove that if $f \in \mathcal{L}\,\mathcal{C}^2$, then f is a Darboux function in Baire class one. Z. Zahorski in [37] considered a hierarchy of classes of functions $f : \mathbb{R} \to \mathbb{R}$. The largest class was equal to the family of all Darboux functions of Baire class one, the smallest, the family of all approximately continuous functions. He also showed how the classes of all derivatives and bounded derviatives fit into the scheme. In this paper we shall define similar classes for functions of two variables in such a way that the fundamental theorems connected with the Zahorski classes will also be true.

Definition 1.5: We say that $x_0 \in \mathbb{R}^2$ is a point of multilateral accumulation (condensation) of a set $P \subset \mathbb{R}^2$ relative to a set B if for every arc $Ł = L(x_0,x_1) \subset B$ x_0 is an accumulation (condensation) point of $P \cap Ł$.

In case $B = \mathbb{R}^2$ we say that x_0 is a point of multilateral accumulation (condensation) of P.

Definition 1.6: We say that the graph of function $f : P \to \mathbb{R}$ is $\mathcal{L}$-connected if the graph of the function $f_{|Ł}$ is connected for each arc $Ł \subset P$.

The class of all functions $f : P \to \mathbb{R}$ in Baire class one having connected graphs ($\mathcal{L}$-connected) is denoted by J_0^P (J^P).

Definition 1.7: Let E be a nonempty set of type F_σ and let $P = \mathbb{R}^2$ or $P \in \mathcal{L}_2$. We say that E belongs to class

M_0^P if every point of E is a point of multilateral accumulation of E relative to the set P;

M_1^P if every point of E is a point of multilateral condensation of E relative to the set P;

M_2^P if for every $x \in E$ and for every arc $Ł = L(x,y) \subset P$ such that $Ł \in \mathcal{L}_2$ $m_Ł(E \cap Ł) > 0$ and for every arc $Ł = L(x,y) \subset P$ such that $Ł \in \mathcal{L}_1 \cup \mathcal{L}_3$ $m_0(E \cap Ł) > 0$;

M_3^P if (1°): there exists a sequence of closed sets $\{K_n\}$ which fulfills the following conditions: $E = \bigcup_n K_n$ and for every $Ł \subset P$ such that $Ł \cap \mathcal{L}_2$ and $Ł \cap E \neq \emptyset$ there exists a sequence of numbers $\{\eta_n\} \subset [0,1)$ such that for each $x \in K_n \cap Ł$ and each $c > 0$ there exists a positive number $\varepsilon(x,c,Ł)$ such that if h,h_1 are different elements of $Ł$ which satisfy conditions: $h \in L_Ł(x,h_1)$,

$$\frac{m_{\text{Ł}}(L_{\text{Ł}}(x,h))}{m_{\text{Ł}}(L_{\text{Ł}}(h,h_1))} < c \quad \text{and} \quad m_{\text{Ł}}(L_{\text{Ł}}(x,h_1)) < \varepsilon(x,c,\text{Ł}), \qquad \text{then}$$

$$\frac{m_{\text{Ł}}(E \cap L_{\text{Ł}}(h,h_1))}{m_{\text{Ł}}(L_{\text{Ł}}(h,h_1))} > \eta_n,$$

and if (2^o): for every $e \in E$ and every arc $\text{Ł} \subset P$ such that $x \in \text{Ł} \in \mathcal{L}_1 \cup \mathcal{L}_3$ $m_0(\text{Ł} \cap E) > 0$;

M_4^P if E fulfills the above conditions (1^o) and (2^o) with the additional assumption that $\eta_n > 0$;

M_5^P if every point of E is a $\mathcal{L}_2$-density point of E relative to P if $P \in \mathcal{L}_2$ or a $\mathcal{L}$-density point of E if $P = \mathbb{R}^2$.

Moreover we assume that the empty set belongs to each of these classes.

Definition 1.8: We say that a function $f : P \to \mathbb{R}$, where $P = \mathbb{R}^2$ or $P \in \mathcal{L}_2$ is in Zahorski class $\mathfrak{M}_i^P$ (for $i = 0,1,\dots,5$) if for every $\alpha \in R$ $E^\alpha(f) \in M_i^P$ and $E_\alpha(f) \in M_i^P$. In case $P = \mathbb{R}^2$ we write $\mathfrak{M}_i^2$.

The first theorem shows that the notions of $\mathcal{L}_1$-density and "standard" density are equivalent.

Definition 1.9: A point $p \in \mathbb{R}^2$ is a $\mathcal{L}_1$-density point of A if and only if it is a density point of A.

The next theorem shows, among other things, that there exists an approximately continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not a Darboux function such that D_f is a singleton.

Theorem 1.10: An interval $I = [a,b]$ is parallel either to the X-axis or to the Y-axis if and only if there does not exist an approximately continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D_f = \{a\}$ and

$$f_{|I}(x) = \begin{cases} 0 & \text{if } x = a, \\ 1 & \text{if } x \in I \setminus \{a\}. \end{cases}$$

The following theorem shows that the classes of functions of two variables under consideration possess properties analogous to the Zahorski classes of functions $f : \mathbb{R} \to \mathbb{R}$. Similar problems for the classes $\mathfrak{M}_0$ and $\mathfrak{M}_1$ (for Darboux $\mathcal{B}$ function) were studied in [1, §4 Theorem 2]. Theorem 1.11 presents some characterizations of the Darboux functions in Baire class one. These characterizations are connected with the notion of multilateral accumulation and condensation points and with $\mathcal{L}$-connectedness of graphs of functions. Throughout this paper the symbol DB_1^P denotes the class of all real Darboux functions in Baire class one which are defined on P.

Theorem 1.11: Let $P = \mathbb{R}^2$ or $P \in \mathcal{L}_2$. Then

$$J_0^P \supset J^P = DB_1^P = \mathfrak{M}_0^P = \mathfrak{M}_1^P \underset{\neq}{\supset} \mathfrak{M}_2^P \underset{\neq}{\supset} \mathfrak{M}_3^P \underset{\neq}{\supset} \mathfrak{M}_4^P \underset{\neq}{\supset} \mathfrak{M}_5^P = \mathcal{L}\,\mathfrak{A}^P.$$

$$C(R^2) \underset{\neq}{\subset} \mathcal{L}\,\mathfrak{A}^2 \underset{\neq}{\subset} \mathfrak{A}^2 = \mathcal{L}_1\,\mathfrak{A}^2.$$

Before we formulate theorems applying the Zahorski classes we adopt the following definitions.

Definition 1.12: Let X,Y be topological spaces.

(a) We say that $f : X \to Y$ possesses the property of Swiatkowski if for every two points x,y such that $f(x) \neq f(y)$ for any arcs $L = L(x,y)$ and $K = L(f(x),f(y))$ and for any open sets $U \subset X$, $V \subset Y$ for which $L \setminus \{x,y\} \subset U$ and $K \setminus \{f(x),f(y)\} \subset V$ there exists a point $z \in U \cap C_f$ such that $f(z) \in V$.

(b) We say that $f : X \to Y$ possesses the strong property of Swiatkowski if for any two points x,y such that $f(x) \neq f(y)$ and for any arcs $L = L(x,y)$ and $K = L(f(x),f(y))$ there exists a point $z \in (L \setminus \{x,y\}) \cap C_f$ such that $z \in K \setminus \{f(x),f(y)\}$.

Remark: It is easy to see that when $f : \mathbb{R} \to \mathbb{R}$ the above definitions are equivalent to the Mańk-Swiatkowski definitions. (See [22].)

Let $Ł \in \mathcal{L}_2 \setminus I^*$. Then $Ł \subset \{(p,\varphi(p)) : a' < p < b'$ and $\varphi \in C_1\}$. Let Φ be the function defined by $\Phi(p) = (p,\varphi(p))$. For arbitrary elements $x,y \in Ł$ we write $x < y$ if and only if $\Phi^{-1}(x) < \Phi^{-1}(y)$. Thus:

$$x - y = \begin{cases} \rho(x,y) & \text{if } x > y, \\ -\rho(x,y) & \text{if } x < y. \end{cases}$$

In case $\mathfrak{L} \in I^*$ we understand $x < y$ in the usual sense (on Y).

Definition 1.13: Let $f : A \to \mathbb{R}$ and let $\mathfrak{L} = L((a_1,b_1),(a_2,b_2)) \in \mathcal{L}_2$ be an arc such that $\mathfrak{L} \subset A$ and $x_0 = (x^0,y^0) \in \mathfrak{L}$. By the $\mathfrak{L}$-derivative $f^{\mathfrak{L}}(x_0)$ of f at x_0 we mean the number

$$f^{\mathfrak{L}}(x_0) = \begin{cases} \dfrac{(f\circ\phi)'(\phi^{-1}(x_0))}{M} & \text{if } \mathfrak{L} \notin I^*, \\ \lim\limits_{\substack{h\to0 \\ (x^0,y^0+h)\in\mathfrak{L}}} \dfrac{f((x^0,y^0+h)) - f((x^0,y^0))}{h} & \text{if } \mathfrak{L} \in I^*, \end{cases}$$

(if such a number exists) where $M = \sup\limits_{[a_1,a_2]} |\phi'|$. The function $f^{\mathfrak{L}}$ will be called the $\mathfrak{L}$-derivative of f.

Theorem 1.14: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Darboux function in Baire class one possessing the strong property of Swiatkowski. Then $f \in \mathfrak{M}_2^2$.

Theorem 1.15: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded $\mathcal{L}$-approximately continuous function. For every arc $\mathfrak{L} \in \mathcal{L}_2$, $f_{|\mathfrak{L}}$ is an $\mathfrak{L}$-derivative.

Theorem 1.16: For every segment $I \subset \mathbb{R}^2$ and every function $f : I \to \mathbb{R}$ the I-derivative f^I is equal to the derivative, f', of the function f considered as a function of one variable.

Theorem 1.17: Let $\mathfrak{L} \in \mathcal{L}_2$. If $F : \mathfrak{L} \to \mathbb{R}$ is an $\mathfrak{L}$-derivative, then $f \in \mathfrak{M}_3^{\mathfrak{L}}$.

Theorem 1.18: Let $\mathfrak{L} \in \mathcal{L}_2$. If $F : \mathfrak{L} \to \mathbb{R}$ is a bounded $\mathfrak{L}$-derivative, then $F \in \mathfrak{M}_4^{\mathfrak{L}}$.

It is well-known that if $\{f_n\}$ is a sequence of Darboux functions of one variable in Baire class one and $f_n \rightrightarrows f$, then f is a Darboux function in Baire class one. This theorem suggests the following question: Is the class $DB_1^{\mathbb{R}^2}$ "uniformly closed"? The answer to this question is contained in the next theorem.

Theorem 1.19: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real functions defined on $\mathbb{R}^2$ such that f_n is a Darboux function in Baire class one. If $f_n \rightrightarrows f$, then f is a Darboux function in Baire class one.

The proof of this theorem follows from Theorem 1.11 and a theorem of E. Kocela. (See [20].) Note that the above theorem is true even if the domain of f is an arbitrary metric space.

In [22] it has been shown that if f_n possesses the property of Swiatkowski and f_n is a Darboux function in Baire class one (for $n = 1,2,\ldots$), and moreover $f_n \rightrightarrows f$, then f possesses the property of Swiatkowski. This theorem is the fundamental theorem of Mańk-Swiatkowski. The next theorem shows that the fundamental theorem of Mańk-Swiatkowski is true also for functions of two variables:

Theorem 1.20: Let f_n, $f : \mathbb{R}^2 \to \mathbb{R}$ and let f_n (for $n = 1,2,\ldots$) be a Darboux function in Baire class one possessing the property of Swiatkowski. If $f_n \rightrightarrows f$, then f possesses the property of Swiatkowski and, of course, f is a Darboux function in Baire class one.

§2. <u>Darboux functions of bounded variation</u>. It is well known that if f is a real, continuous function defined on the interval $[a,b]$ and if N_f denotes the Banach indicatrix of f, then N_f is a function in the second Baire class and the variation of f, $\overset{b}{\underset{a}{V}}(f)$, is equal to $\int_{-\infty}^{+\infty} N_f(y)dy$. The notion of Banach indicatrix can be generalized in the following natural way. (See [33, p. 217].)

Definition 2.1: The Banach indicatrix of a function $f : E \to Y$ with respect to a set $D \subset E$ is a function $N_f^D : Y \to \overline{\mathbb{R}}$ defined as follows:

$$N_f^D(p) = \begin{cases} \operatorname{card}(f^{-1}(p) \cap D) & \text{if } f^{-1}(p) \cap D \text{ is a finite set,} \\ +\infty & \text{if } f^{-1}(p) \cap D \text{ is an infinite set.} \end{cases}$$

In [31, Theorem 2], T. Šalat has proved that if X is a locally connected Hausdorff space with a countable basis and if $f : X \to \mathbb{R}$ is a connected transformation (i.e. f preserves connectedness), then N_f is in the second Baire class. The next theorem shows that for Darboux functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ Šalat's theorem is false.

Theorem 2.2: There exists a Darboux function $f : \mathbb{R}^2 \to \mathbb{R}^2$ for which the Banach indicatrix N_f^I is a nonmeasurable function where $I = [(0,0),(1,0)]$.

The above theorem suggests the following question: Under what additional assumptions is the Banach indicatrix N_f^D a measurable function? Of course the answer is positive if $m_2(f(D)) = 0$. The theorem below presents a more interesting, positive answer:

Theorem 2.3: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a Darboux function, L-regular*) with respect to the closed and locally connected set D. Then the Banach indicatrix N_f^D is a measurable function.

Now we pass to the variation of a Darboux function. First we assume the following definition. (See [33, Definition 2, p. 217].)

Definition 2.3: Let $f : E \to \mathbb{R}^2$ $(E \subset \mathbb{R}^2)$ be a Darboux function such that the Banach indicatrix N_f^D $(D \subset E)$ is a measurable function. The function f is said to be of bounded variation in D (in the Banach sense) if

$$\int_{\mathbb{R}^2} N_f^D(p)\, dm_2 < \infty.$$

As we know if $f : I \to \mathbb{R}$ is a Darboux function such that $D_f \neq \emptyset$, then the variation $\overset{b}{\underset{a}{V}}(f) = +\infty$. Simple examples show that there exist discon-

*) We say that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is L-regular with respect to a set $A \subset \mathbb{R}^2$ if for every open (in A) set W and every component K of $f(W)$ $\operatorname{Fr} K \in L$ where L denotes the class of all sets A_0 for which $m_2(A_0) = 0$).

tinuous Darboux functions $f : I \times I \to \mathbb{R}^2$ for which the variation $\underset{I\times I}{V}(f) < +\infty$. Moreover, we can prove:

Theorem 2.4: In the space of bounded, Darboux functions of bounded variation $f : I \times I \to \mathbb{R}^2$ (with the usual metric in the space of bounded functions) the set of discontinuous functions is a dense set of cardinality 2^c.

§3. __On some extension of Darboux functions.__

In the theory of continuous functions many important theorems are concerned with extensions of functions. In this part we shall consider some problems relating to extending Darboux functions $f : F \to \mathbb{R}^2$ ($F \subset \mathbb{R}^2$).

Theorem 3.1: Let F be an arbitrary closed convex subset of $\mathbb{R}^2$ and let $f : F \to \mathbb{R}^2$ be a Darboux function. Then for every $\varepsilon > 0$ there exists a Darboux function $f^* : \mathbb{R}^2 \to \mathbb{R}^2$ such that $C_{f^*} = C_f$ and f^* is an ε-extension of f.

Observe that if $\varepsilon = 0$, then the above theorem is false.

Simple examples show that in the last theorem the assumption that F is a closed, convex subset of $\mathbb{R}^2$ cannot be replaced by the assumption that F is a continuum. Moreover, according to Mazurkiewicz's theorem [23], we deduce:

Theorem 3.2: The set of all continuums F for which there exists a Darboux function $f : F \to \mathbb{R}^2$ which does not possess a Darboux 1-extension is dense in the exponential space of $I \times I$ (with Hausdorff metric). (See [8].)

On the other hand we can prove:

Theorem 3.3: Let F be such a subset of $\mathbb{R}^2$ for which there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(F)$ is a closed, convex set and let $f : F \to \mathbb{R}^2$ be a Darboux function. Then there exists a Darboux extension f^* of f over $\mathbb{R}^2$ such that $C_f = C_{f^*}$.

Theorem 3.4: Let $F = F_1 \cup F_2 \subset I \times I$ where F_1, F_2 are disjoint, closed, convex sets and let $f : F \to \mathbb{R}^2$ be a Darboux function. Then there exists a Darboux extension f^* of F over $\mathbb{R}^2$ such that $C_f = C_{f^*}$.

We shall close this section with a theorem involving the extension of Darboux functions in Baire class $\alpha (0 \leq \alpha < \Omega)$. The fact that f is in Baire class α is denoted by $f \in B_\alpha$.

Theorem 3.5: Let F be a closed, convex subset of $\mathbb{R}^2$ and let $f : F \to \mathbb{R}^2$ be a Darboux function such that $f \in B_\alpha$ $(0 \leq \alpha < \Omega)$. Then there exists a Darboux 0-extension f^* of f over $\mathbb{R}^2$ such that $f^* \in B_\alpha$.

§4. <u>On local characterizations of Darboux functions</u>.

In this paper published in 1965, ([2]), A.M. Bruckner and J.G. Ceder introduced the notion of a Darboux point of a function $f : \mathbb{R} \to \mathbb{R}$. Basic properties of Darboux points were studied in papers [3], [11], [15], [16], [17], [21], [29] and [31]. In this part we study generalizations of the notion of a Darboux point in such a way that theorems about local characterizations of Darboux transformations in arbitrary topological spaces will hold.

Let $f : X \to Y$ where X and Y are arbitrary topological spaces.

Definition 4.1: We say that a point $x_0 \in X$ is a Darboux point of the first kind (of the function f) if for every arc $\mathcal{L} = L(x_0, a)$ the following conditions are fulfilled:

1°) if $\overline{f(L_{\mathcal{L}}(x_0,p))} = Y$ for every element $p \in \mathcal{L} \setminus \{x_0\}$, then there exists a point $p_0 \in \mathcal{L} \setminus \{x_0\}$ such that $f(L_{\mathcal{L}}(x_0,p_0))$ is a connected set;

2°) if K is a set such that for some net $\{x_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{L}$ for which $x_0 \in \lim_{\sigma\in\Sigma} x_\sigma$ K quasi-cuts $f(\mathcal{L}) \cup \operatorname{acp}_{\sigma\in\Sigma} f(x_\sigma)$*) between the sets

*) By $\operatorname{acp}_{\sigma\in\Sigma} f(x_\sigma)$ we denote the set of all accumulation points of $\{x_\sigma\}_{\sigma\in\Sigma}$.

$\{f(x_0)\}$ and $\{f(x_\sigma) : \sigma \in \Sigma\} \cup \underset{\sigma\in\Sigma}{\operatorname{acp}} f(x_\sigma)$, then
$K \cap f(L_Ł(x_0,x_\sigma)) \neq \emptyset$, for every $\sigma \in \Sigma$;

3°) if for some net $\{x_\sigma\}_{\sigma\in\Sigma} \subset Ł$ for which $x_0 \in \lim_{\sigma\in\Sigma} x_\sigma$ $Y \setminus f(Ł)$ quasi-cuts $f(Ł)$ into sets A and B between the sets $\{f(x_0)\}$ and $\{f(x_\sigma) : \sigma \in \Sigma\}$ in such a way that $\overline{A} \cap \overline{B} \neq \emptyset$, then $\overline{A} \cap \overline{B}$ is of type G_δ in the subspace $\overline{A \cup B}$ of Y.

Definition 4.2: We say that a point $x_0 \in X$ is a Darboux point of the second kind (of the function f) if for every arc $Ł = L(x_0,a)$ conditions 1°) and 2°) of Definition 4.1 are satisfied.

Definition 4.3: We say that a point $x_0 \in X$ is a Darboux point of the third kind (of the function f) if for every arc $Ł = L(x_0,a)$ the following condition is fulfilled:

if K is a set such that for some net $\{x_\sigma\}_{\sigma\in\Sigma} \subset Ł$ for which $x_0 \in \lim_{\sigma\in\Sigma} x_\sigma$ K cuts Y between $\{f(x_0)\}$ and $\{f(x_\sigma) : \sigma \in \Sigma\} \cup \underset{\sigma\in\Sigma}{\operatorname{acp}} f(x_\sigma)$, then $K \cap f(L_Ł(x_0,x_\sigma)) \neq \emptyset$ for every $\sigma \in \Sigma$.

It is easy to see that if x_0 is a Darboux point of the first (second) kind of f, then x_0 is a Darboux point of the second (third) kind of f.

Theorem 4.4: Let $f : X \to Y$ where X and Y are arbitrary topological spaces. If $x_0 \in C_f$, then x_0 is a Darboux point of the first kind of a function f. (Of course, then x_0 is a Darboux point of the second and of the third kind of f.)

The next theorem shows that in case $f : \mathbb{R} \to \mathbb{R}$ the above definitions are equivalent to the usual definition of a Darboux point.

Theorem 4.5: Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Then the following conditions are equivalent:

(i) x_0 is a Darboux point of the first kind of f,

(ii) x_0 is a Darboux point of the second kind of f,

(iii) x_0 is a Darboux point of the third kind of f,

(iv) x_0 is a Darboux point (in the usual sense).

A. Császár in paper [5] has shown that a real function f is Darboux on I if and only if for every $x \in I$ x is a Darboux point of f. (Also see [2] and [9].) The next theorem shows that a similar result is also true for transformations in topological spaces.

Theorem 4.6: Let $f : X \to Y$ where X and Y are arbitrary topological spaces. Then f is a Darboux transformation if and only if every point $x \in X$ is a Darboux point of the first kind of f.

One can show that in the previous theorem the necessary condition cannot be replaced by the condition "every point $x \in X$ is a Darboux point of the second kind of f".

Theorem 4.7: There exist topological spaces X and Y and a transformation $f : X \to Y$ such that $D_f = \{x_0\}$, x_0 is a Darboux point of the second kind of f, and f is not a Darboux transformation.

In paper [27], T. Radaković has defined a new class of functions. The next definition presents the generalization of this notion to transformations whose domains and ranges are arbitrary topological spaces.

Definition 4.8: Let X,Y be arbitrary topological spaces. We say that $f : X \to Y$ is a Darboux transformation in the sense of Radaković, if $\overline{f(Ł)}$ is a connected set for each arc $Ł \subset X$.

Theorem 4.9: Let $f : X \to Y$ where X and Y are arbitrary topological spaces. If every element $x \in X$ is a Darboux point of the second kind of f, then f is a Darboux transformation in the sense of Radaković.

It is easy to see that there exists a Darboux function in the sense of Radaković $f : \mathbb{R} \to \mathbb{R}$ for which any point of $\mathbb{R}$ is not a Darboux point of the second kind of f.

The next theorem can be interpreted as a local characterization of Darboux transformations in B^1 where $f \in B^1$ means $C_{f|F} \neq \emptyset$ for every closed set F.

Theorem 4.10: Let X be a T_2-space and Y be a T_1-space. Let $f : X \to Y$ be a transformation belonging to class B^1. Then f is a Darboux transformation if and only if every point $x \in X$ is a Darboux point (of the second kind) of f.

One can show that in the above theorem the necessary condition cannot be replaced by the condition "every point $x \in X$ is a Darboux point of the third kind of f".

Comments and Theorem 5.1 from paper [2] show that the notion of a Darboux point is presented in such a way that the local characterization of a Darboux function holds. The next definition presents the concept of a D-point which is not equivalent to the notion of a Darboux point in the sense of A. Bruckner and J. Ceder, but for which the local charcterization of a Darboux function does hold.

Definition 4.11: Let $f : X \to Y$ where X, Y are arbitrary topological spaces. We say that an element $x_0 \in X$ is a D-point of f if $x_0 \in C_f$ or the following conditions are fulfilled:

1. If there exists a point $p \in X$ such that $f(L(x_0,p)) = Y$, then Y is a connected space.

2. For an arbitrary arc $L(x_0,a)$ there exists an element $y \in L(x_0,a) \setminus \{x_0\}$ such that for every $z \in L_{L(x_0,a)}(x_0,y)$ if K cuts Y between $\{f(x_0)\}$ and $\{f(z)\}$, then $K \cap f(L_{L(x_0,a)}(x_0,z)) \neq \emptyset$.

It is not hard to verify that if f is an arbitrary transformation and if x_0 is an arbitrary element from the domain of f, then:

$$\begin{array}{ccc}
\text{x_0 is a Darboux point of the first kind of f} & \Leftarrow & \text{x_0 is a point of continuity of f} \\
\Downarrow & & \Downarrow \\
\text{x_0 is a Darboux point of the second kind of f} & & \text{x_0 is a D-point of f} \\
\Downarrow & & \Downarrow \\
\end{array}$$

$$\text{x_0 is a Darboux point of the third kind of f.}$$

The reverse implications are not true.

Theorem 4.12: Let $f : X \to Y$ where X is an arbitrary topological space and Y is T_5-space. (See [7].) Then f is a Darboux transformation if and only if every element $x \in X$ is a D-point of f.

We shall close this paper with a theorem which presents an application of the notion of a Darboux point. This application is connected with results presented in papers [13], [14], [17], [19], [24], [35] and [34].

Theorem 4.13: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a closed function. Then $x_0 \in C_f$ if and only if x_0 is a Darboux point of the third kind of f.

Note that in the preceding theorem the assumption that "f is a closed function" can be replaced by the assumption that "f is a closed function at x_0". (See [26, Definition 3].) Of course the condition "x_0 is a Darboux point of the third kind" can be replaced by "x_0 is a Darboux point of the first kind", "x_0 is a Darboux point of the second kind" or "x_0 is a D-point of f".

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