INROADS Real Analysis Exchange Vol. 11 (1985-86)

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ON PROJECTIONS OF PLANAR SETS

If $a \in R$ and $A \subseteq R^2$, we say that the projection of A in direction a is $\{c : gr(y = ax + c) \cap A \neq \emptyset\}$. Here $gr(y = ax + c)$ denotes the graph of the function $y = ax + c$. The c-projection of A in direction a is $\{c : dom[gr(y = ax + c) \cap A]$ is of second category$\}$. The m-projection (m^*-projection) of A in direction a is $\{c : m(dom[gr(y = ax + c) \cap A]) > 0\}$ ($\{c : m^*(dom[gr(y = ax + c) \cap A]) > 0\}$). Here m denotes Lebesgue measure in R and m^* denotes outer Lebesgue measure in R. The following question was formulated in [1]: "Does there exist a linear set A of second category such that the projection of $A \times A$ onto each line has empty interior?" The partial solution if Martin's Axiom is assumed was provided in [7]. The solution under CH was given in [2] by Davies. Observe that the proof of the theorem of Davies does not really require CH (change "$< \omega_1$" to "τ", "only countable" to "fewer than τ" and recall that uncountable closed sets must have cardinality τ). Thus we obtain Proposition 1.

Proposition 1. There exists a second category set A such that the projection of $A \times A$ in each direction does not contain an interval.

In Proposition 2 we construct a c-Lusin set L for which every c-projection of $L \times L$ in direction $a \neq 0$ is equal to R (under MA). If L is of the first category, then, as is well known, any c-projection of $L \times L$ is empty. In Proposition 3 we construct a strong, first category set S for which every m^*-projection of $S \times S$ in direction $a \neq 0$ is equal to R (under MA). Let $C \subseteq R - \{0\}$ be a set of cardinality less than that of the continuum. In Proposition 4 we construct sets A and B of the second category such that every projection of $A \times B$ in direction $c \in C$ equals R and every c-projection of $A \times B$ in direction $c \in C$ is empty (under MA). We use a technique due to Erdös, Kunen and Mauldin [3].

We use the following notation. If $A,B \subseteq R$, then $A+B = \{a+b : a \in A, b \in B\}$, $A \cdot B = \{ab : a \in A, b \in B\}$, $A-B = \{a-b : a \in A, b \in B\}$ and $A \setminus B =$

$\{x : x \in A$ and $x \notin B\}$. Let A(m) stand for the proposition that the union of less than continuum many measure zero sets has measure zero. Let U(m) mean that every set of reals of cardinality less than τ has measure zero. A(c) and U(c) are defined similarly with meager replacing measure zero [5]. (A(c) is sometimes referred to as the Strong Baire Category Theorem (SBCT).)

Recall that the following implications hold:

$$\begin{array}{ccc} & & A(m) \longrightarrow U(m) \\ CH \longrightarrow MA & & \\ & & A(c) \longrightarrow U(c) \qquad \text{(see [9])} \end{array}$$

A set $X \subseteq R$ is a c-Lusin (c-Sierpinski) set iff $|X \cap M| < \tau$ for each meager (measure zero) set $M \subseteq R$. (See [6].) Recall that under MA every c-Lusin set X has measure zero. Indeed if G is a comeager set of measure zero (see [8], Corollary 1.7) then $F = R \setminus G$ is meager and $X = (X \cap G) \cup (X \cap F)$. Since $|X \cap F| < \tau$ and U(m) holds we have $m(X \cap F) = 0$. Thus $m(X) = m(X \cap G) + m(X \cap F) = 0$. Similarly under MA every c-Sierpinski set is meager. In Propositions 2 and 4 we assume A(c). In Proposition 3 we assume A(m).

Proposition 2. Assume A(c). There is a c-Lusin set L such that every c-projection of $L \times L$ in direction $a \neq 0$ is equal to R. If MA holds, then the m-projection of $L \times L$ in each direction is empty. If CH holds, then L is a Lusin set.

Proof. List all meager F_σ sets: F_α, $\alpha < \tau$. List all dense G_δ sets: G_α, $\alpha < \tau$. List all (non-horizontal and non-vertical) lines in R^2: k_α, $\alpha < \tau$, with the additional property that each line appears τ-times. Let

$H_\alpha = \bigcap_{\beta \leq \alpha} G_\beta \setminus \bigcup_{\beta \leq \alpha} F_\beta$ and $K_\alpha = \{y : \exists x \in H_\alpha \ (x,y) \in k_\alpha\}$ for each $\alpha < \tau$.

Observe that the sets H_α, K_α and $H_\alpha \cap K_\alpha$ are comeager. At level α choose $y_\alpha \in H_\alpha \cap K_\alpha$ and $x_\alpha \in H_\alpha$ such that $(x_\alpha, y_\alpha) \in k_\alpha$. Let $L = \bigcup_{\alpha<\tau} \{x_\alpha, y_\alpha\}$. Then L has the desired properties. Clearly, $|L \cap M| < \tau$ for each meager $M \subseteq R$. Notice that $G_\beta \cap \mathrm{dom}(L \times L \cap k_\alpha) \neq \emptyset$ for each $\alpha,\beta < \tau$. Indeed since $|\{\gamma < \tau : k_\alpha = k_\gamma\}| = \tau$, we have $\gamma > \beta$ such that

$k_\gamma = k_\alpha$. Then $x_\gamma \in G_\beta \cap \mathrm{dom}(L \times L \cap k_\gamma) = G_\beta \cap \mathrm{dom}(L \times L \cap k_\alpha)$. Thus $\mathrm{dom}(L \times L \cap k_\alpha)$ is of the second category. Hence the c-projection of $L \times L$ in the direction k_α is equal to R.

Assume that MA holds. Then $m(L) = 0$ and every m-projection of $L \times L$ is empty.

A set of reals X has the strong first category property iff for every set H of measure zero there exists a real x such that $(x + X) \cap H = \emptyset$. (See [6].) The next proposition can be proved in a similar fashion to Proposition 2.

Proposition 3. Assume A(m). There is a c-Sierpinski set S which has the strong first category property and such that every m^*-projection of $S \times S$ in direction $a \neq 0$ is equal to R. If MA holds, then the c-projection of $S \times S$ in each direction is empty. If CH holds, then S is a Sierpinski set.

Proof. List all measure zero G_δ sets: G_α, $\alpha < \tau$. List all full measure F_σ sets: F_α, $\alpha < \tau$. List all (non-horizontal and non-vertical) lines in R^2: k_α, $\alpha < \tau$, with the additional property that each line appears τ-times. At level α choose x_α, y_α and z_α such that $z_\alpha \in R \setminus \bigcup_{\beta<\alpha} [(G_\alpha - x_\beta) \cup (G_\alpha - y_\beta)]$, and $x_\alpha, y_\alpha \in \bigcap_{\beta\leq\alpha} F_\beta \setminus \bigcup_{\beta\leq\alpha} [G_\beta \cup (G_\beta - z_\beta)]$, and $(x_\alpha, y_\alpha) \in k_\alpha$. Let $S = \bigcup_{\alpha<\tau} \{x_\alpha, y_\alpha\}$. Then S is a c-Sierpinski set. Thus $(z_\alpha + S) \cap G_\alpha = \emptyset$ for each $\alpha < \tau$ and S has the strong first category property. Similarly as in Proposition 2 we can show that every m^*-projection of $S \times S$ in direction $a \neq 0$ equals R. If MA holds, then S is meager. Thus every c-projection of $S \times S$ is empty.

In [3] the following theorem is proved:

"Theorem 7. Suppose that the union of less than continuumly many meager subsets of R is meager. There are subsets G_1 and G_2 of R both of which are subspaces of R over the field of rationals both of which meet every meager set in a set of cardinality less than τ and such that $G_1 \cap G_2 = \{0\}$ and $G_1 + G_2 = R$. (Of course, if every subset of R with cardinality less than τ has measure zero, then G_1 and G_2 both have measure zero. If CH holds, then G_1 and G_2 are both Lusin sets.)"

Notice that every projection of $G_1 \times G_2$ in direction $q \in Q - \{0\}$ is equal to R and every c-projection of $G_1 \times G_2$ in direction $q \in Q - \{0\}$ is empty. Indeed $R = G_1 + G_2 = G_2 - qG_1$ for each $q \in Q - \{0\}$. Suppose that $c = y-qx = y_1-qx_1$ for $q \in G - \{0\}$, $x,x_1 \in G_1$ and $y,y_1 \in G_2$. Since $G_1 \cap G_2 = \{0\}$, $x = x_1$ and $y = y_1$. Hence $dom[gr(y = qx + c) \cap G_1 \times G_2]$ is of the first category and the c-projection of $G_1 \times G_2$ in direction q is empty.

The next proposition can be verified in a similar fashion.

Proposition 4. Let C be a subset of $R - \{0\}$ of cardinality less than that of the continuum. There are c-Lusin sets $A,B \subseteq R$ both of which are subspaces of R over the field Q(C) (here Q(C) is the extension of Q by elements of C) and such that $A \cap B = \{0\}$, every projection of $A \times B$ in direction $a \in C$ is equal to R and every c-projection of $A \times B$ in direction $a \in C$ is empty. (If CH holds, then A and B are both Lusin sets.)

Observe that if C is the projection of $A \times B$ in direction a, then $C = B-aA$. Thus by Proposition 1 we have that there exists a set A of second category such that $A-aA$ has empty interior. Let $A \underset{C}{-} B = \{c : \{a \in A: \exists\, b \in B \;\; c = a-b\}$ is of second category$\}$. From Proposition 2 it follows that there exists (under CH) a Lusin set L such that $L \underset{C}{-} L = R$. The first result in the spirit of this proposition was given by Sierpinski in [10] where he showed (under CH) that there is a Lusin set L such that $L-L = r$. In a similar spirit E. Grzegorek has recently shown in ZFC that there exists an universal measure zero (always of the first category) set A such that $m^*(A+A) > 0$ (A+A is second category) [4]. If ZFC is consistent, then ZFC + "$c = 2^{\omega_1}$" + "every universal measure zero set has cardinality at most ω_1" + U(c) is consistent. (See [5] and [6].) Hence it is unprovable in ZFC that there exists an universal measure zero set A such that $A \underset{C}{+} A \neq \emptyset$.

Observe that $A \underset{C}{-} B = R$ for every comeager set A and second category set B. Indeed $\{a \in A: \exists\, b \in B \;\; x = a-b\}$ is equal to $A \cap (x+B)$ for each $x \in R$.

I would like to thank the referee for very helpful suggestions concerning the contents of this paper. (The proofs of Propositions 1 and 2 were originally more complicated.)

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Received May 2, 1985