Real Analysis Exchange Vol. 11 (1985-86)

Jerzy Niewiarowski, Institute of Mathematics, University of Lodz,
ul. Banacha 22, 90-238 Lodz, POLAND.

MONOTONE SECTIONS OF FUNCTIONS OF TWO VARIABLES

We introduce the following notation.

Let $\phi : I \to R$ (where $I = [0,1]$).

If a function ϕ has a property P, we denote this fact by $P(\phi)$.

Let $f : I \times I \to R$. Then for each $x \in I$ we consider $f_x(y) = f(x,y)$ as a function of y and for each $y \in I$ we consider $f^y(x) = f(x,y)$ as a function of x.

We put

$$A_x(f,P) = \{x;P(f_x)\} \quad \text{and} \quad A_y(f,P) = \{y;P(f^y)\}.$$

Let $A_1 \subset I$ and $A_2 \subset I$. We investigate conditions on the sets A_1 and A_2 under which there exists a function f such that $A_1 = A_x(f,P)$ and $A_2 = A_y(f,P)$, where P is a certain fixed property such as "nondecreasing", "increasing", "nondecreasing and continuous", "increasing and continuous", "of bounded variation".

Then we construct a function fulfilling these conditions. At first we suppose that P means "nondecreasing".

Theorem 1. Let $A_1,A_2 \subset I$. Then there exists a function $f(x,y)$ defined on $I \times I$ such that $A_1 = A_x(f,P)$ and $A_2 = A_y(f,P)$ if and only if

1° $I \neq A_1$ and $I \neq A_2$ or

2° $A_1 = A_2 = I$ or

3° $A_1 = I$, $A_2 \neq I$ and $\operatorname{card}(\bar{A}_2 - A_2) \leq \aleph_0$ or
$A_2 = I$, $A_1 \neq I$ and $\operatorname{card}(\bar{A}_1 - A_1) \leq \aleph_0$.

Proof. Sufficiency. If condition 1° or 2° is fulfilled, we define the function $f(x,y)$ in the following way:

$$f(x,y) = \begin{cases} (x+1)(y+1) & \text{for } (x,y) \in A_1 \times [0,1] \cup [0,1] \times A_2 \qquad \text{(I)} \\ -(x+1)(y+1) & \text{for the remaining } (x,y) \in I \times I. \qquad \text{(II)} \end{cases}$$

If $A_1 = A_2 = I$, then, obviously, only (I) is valid, and if $A_1 = A_2 = \emptyset$, then all points are remaining, so we use only (II). It is clear that the function $f(x,y)$ fulfills all the the required conditions. Let us suppose that the first part of condition 3° is fulfilled. Then $I^o = (I^o - \bar{A}_2) \cup ((\bar{A}_2 - A_2) \cap I^o) \cup (A_2 \cap I^o)$ where $I^o = (0,1)$. Let $Z = (\bar{A}_2 - A_2) \cap I^o$. We can write down all elements of the set Z as $\{y_n\}$ because Z is finite or countable. Let $G = I^o - \bar{A}_2 = \bigcup_n (\alpha_n,\beta_n)$ where (α_n,β_n) are components of G. The notation $\bigcup_n (\alpha_n,\beta_n)$ means that the union is finite or countable.

We put

$$g(y) = y + \sum_{y_i<y} \frac{1}{2^i} \qquad \text{for} \quad y \in I^o.$$

Next we define a function $f_1(x,y)$ on the set $I^o \times I^o$ by the formula

$$f_1(x,y) = \begin{cases} x \cdot g(y) & \text{for } x \in I^o - \{\frac{1}{2}\}, \quad y \in I^o. \\ \frac{1}{2} \lim\limits_{\eta \to y^+} g(\eta) & \text{for } x = \frac{1}{2}, \quad y \in I^o. \end{cases}$$

Let $\gamma_n = \lim\limits_{\eta \to \alpha_n^+} f_1(\frac{1}{2},\eta)$, $\delta_n = \lim\limits_{\eta \to \beta_n^-} f_1(\frac{1}{2},\eta)$. Let $h_n(y)$ be any increasing and continuous function defined on $[\alpha_n,\beta_n]$ such that $h_n(\alpha_n) = \gamma_n$, $h_n(\beta_n) = \delta_n$ and $h_n(y) < f_1(\frac{1}{2},y)$ for $y \in (\alpha_n,\beta_n)$.

Now we define a function $f(x,y)$ on the set $I^o \times I^o$ by

$$f(x,y) = \begin{cases} h_n(y) & \text{for } x = \frac{1}{2}, \quad y \in (\alpha_n,\beta_n) \\ f_1(x,y) & \text{for the ramaining } (x,y) \in I^o \times I^o. \end{cases} \qquad \text{(III)}$$

It is not difficult to extend the function $f(x,y)$ to $I \times I$ in order to obtain a function fulfilling all of the required conditions.

Necessity. We suppose that $A_1 = I$, $A_2 \subset I$ and $\mathrm{card}(\bar{A}_2 - A_2) > \aleph_0$. The case $A_1 \subset I$, $A_2 = I$ and $\mathrm{card}(\bar{A}_1 - A_1) > \aleph_0$ is analogous. We assume that we can construct a function $f(x,y)$ such that $A_1 = A_x(f,P)$ and $A_2 = A_y(f,P)$. The set of all limit points of the set A_2 from both the left and the right side which do not belong to the set A_2 is denoted by B. Of course, $\mathrm{card}\, B > \aleph_0$. If $y_0 \in B$, then there exist points x_0, $x_1 \in I$ such that

$$(1) \qquad x_0 < x_1 \quad \text{and} \quad f^{y_0}(x_0) > f^{y_0}(x_1).$$

We shall show that the function $f_{x_0}(y)$ is not continuous at y_0. Suppose that the function $f_{x_0}(y)$ is continuous at y_0. We consider a sequence $\{y_n\}_{n \in N}$ such that $y_n \in A_2$ and $y_n \to y_0$. Of course, $f(x_0,y_n) \leq f(x_1,y_n)$.

Hence $f(x_0,y_0) \leq f(x_1,y_0)$. We have contradicted (1).

Let $x_2 \in (x_0,x_1)$. If $f(x_2,y_0) > f(x_1,y_0)$, then, as above, the function $f_{x_2}(y)$ is not continuous at y_0.

Let $f(x_2,y_0) \leq f(x_1,y_0)$. We suppose that the function $f_{x_2}(y)$ is continuous at y_0. From (1) we obtain $f(x_2,y_0) < f(x_0,y_0)$. There exists $\delta > 0$ such that for each $y \in (y_0,y_0 + \delta)$, $f(x_2,y) < f(x_0,y_0)$.

There exists $y_1 \in A_2 \cap (y_0,y_0 + \delta)$ such that $f(x_0,y_1) > f(x_2,y_1)$. We have a contradiction because $y_1 \in A_2$. We obtain that all functions $f_x(y)$ for $x \in [x_0,x_1)$ are not continuous at y_0. So for $y_0 \in B$ we have found an interval of discontinuity. With each $y \in B$ we associate exactly one such interval. There eixsts a point $x \in I$ which belongs to an uncountable family of intervals. Hence the set of points of discontinuity of the function $f_x(y)$ is uncountable. We obtain a contradiction because this function is nondecreasing.

Remark 1. If $A_1 = I$ and if there exists a function $f(x,y)$ such that $A_1 = A_x(f,P)$, $A_2 = A_y(f,P)$, then A_2 belongs to the class G_δ.

Proof. $I^o = (I^o \cap A_2) \cup (I^o \cap (\bar{A}_2 - A_2)) \cup (I^o - \bar{A}_2)$. Hence A_2 is a set of type G_δ because $I^o \cap (\bar{A}_2 - A_2)$ is a set of type F_σ.

Definition 1. We say that a set B fulfills condition (*) with respect to a set A_2 if and only if there exists a sequence of sets $\{B_n\}$ such that $B = \bigcup_n B_n$ and for every n $\operatorname{card}(A_2 \cap B_n^c) \leq \aleph_0$ (B_n^c denotes the set of all points of condensation of the set B_n).

Remark 2. If a set B does not fulfill condition (*) with respect to A_2 and Z is a finite or countable set, then the set $B-Z$ does not fulfill condition (*) with respect to A_2.

Lemma 1. If a set $\bar{A}_2 - A_2$ fulfills condition (*) with respect to A_2 where $A_2 \subset I$, then A_2 belongs to the intersection of the classes $F_{\sigma\delta}$ and $G_{\delta\sigma}$.

Proof. Let $B = \bar{A}_2 - A_2$, $B = \bigcup_n B_n$ and for every n $\operatorname{card}(A_2 \cap B_n^c) \leq \aleph_0$. For every n let $Z_n = A_2 \cap B_n^c$ and $A_2(n) = (A_2 - Z_n) \cap I^o$. Let G_n denote the union of all maximal neighborhoods $P^{(n)}$ of points of the set $A_2(n)$ such that $\operatorname{card}(P^{(n)} \cap B_n) \leq \aleph_0$. Then $\operatorname{card}(G_n \cap B_n) \leq \aleph_0$ and $\bigcap_n A_2(n) \subset \bigcap_n G_n$, but $\bigcap_n A_2(n) = I^o \cap \bigcap_n (A_2 - Z_n) = I^o \cap (A_2 - Z)$ where $Z = \bigcup_n Z_n$, $\operatorname{card} Z \leq \aleph_0$

From the following inclusion

$$\bigcap_n G_n \cap \bigcup_n B_n \subset \bigcup_n (G_n \cap G_n)$$

we obtain that

$$\operatorname{card}(\bigcap_n G_n \cap G) \leq \aleph_0.$$

We have

$$\bigcap_n G_n \cap A_2 = (A_2 - Z) \cap I^o$$

and

$$(A_2 - Z) \cap I^o = (\bigcap_n (G_n \cap I^o) - \bigcap_n G_n \cap (I^o - \bar{A}_2)) - \bigcap_n G_n \cap B.$$

The set $(A_2 - Z) \cap I^o$ is the intersection of a set of type G_δ and a set of type F_σ. Hence A_2 belongs to the intersection of classes $F_{\sigma\delta}$ and $G_{\delta\sigma}$.

Lemma 2. If for each $y \in B \subset I \subset OY$ the interval $(\alpha_y, \beta_y) \subset I \subset OX$ is nondegenerate, then there exists a sequence of sets $\{B_n\}_{n\in N}$ and a sequence a nondegenerate intervals $\{P_n\}_{n\in N}$, $P_n \subset I \subset OX$ such that $B = \bigcup_n B_n$ and for every n $\bigcap_{y\in B_n} (\alpha_y, b_y) \supset P_n$.

Proof. Let $\{P_n\}_{n\in N}$ be the sequence of all intervals with rational end-points such that $P_n \subset I$. Then the sequence of sets B_n is defined by

$$B_n = \{y \in B;\ (\alpha_y, \beta_y) \supset P_n\}.$$

Now we suppose that P means "increasing".

Theorem 2. There exists a function $f(x,y)$ on $I \times I$ such that $A_1 = A_x(f,P)$ and $A_2 = A_y(f,P)$ if and only if

1° $I \neq A_1 \subset I$ and $I \neq A_2 \subset I$ or
2° $A_1 = A_2 = I$ or
3° $A_1 = I$, $A_2 \subset I$ and $\bar{A}_2 - A_2$ fulfills condition (*) with respect to A_2 or

$A_2 = I$, $A_1 \subset I$ and $\bar{A}_1 - A_1$ fulfills condition (*) with respect to A_1.

Proof. Sufficiency. If condition 1° or 2° is fulfilled, we define the function $f(x,y)$ by (I) or II). (See the proof of Theorem 1.) Now we suppose that $A_1 = I$, $I \neq A_2 \subset I$ and $\bar{A}_2 - A_2$ fulfills condition (*) with respect to A_2. Then by Lemma 1 there exists a set H of type G_δ such that $I^\circ \cap (A_2 - Z) \subset H$, $Z \subset A_2$, card $Z \leq \aleph_0$. Therefore $A = \{y_n\}_{n\in N}$ and card $(H \cap B) \leq \aleph_0$. Let $\tilde{B} = H \cap B$, $H \bigcap_{i=o}^{\infty} G_i$ where $\{G_i\}$ is a non-increasing sequence of open sets. Put $G_i = \bigcup_n (\alpha_n^{(i)}, \beta_n^{(i)})$, where $(\alpha_n^{(i)}, \beta_n^{(i)})$ are the components of the set G_i^n, and $\{y_i^{(n)}\}_{i\in N} =$

$(\alpha_n^{(o)}, \beta_n^{(o)}) \cap \tilde{B}$. For every n let $\{z_i^{(n)}\}_{i\in N}$ be a sequence such that

$$\sum_i z_i^{(n)} = \frac{\beta_n^{(o)} - \alpha_n^{(o)}}{2} .$$

For every n define a function $h_n(y)$ for $y \in (\alpha_n^{(o)}, \beta_n^{(o)})$ by

$$h_n(y) = \begin{cases} \sum_{y_i^{(n)} < y} z_i^{(n)} & \text{if } (\alpha_n^{(o)}, \beta_n^{(o)}) \cap \tilde{B} \neq \emptyset \\ \dfrac{\beta_n^{(o)} - \alpha_n^{(o)}}{2} & \text{if } (\alpha_n^{(o)}, \beta_n^{(o)}) \cap \tilde{B} = \emptyset. \end{cases}$$

Now we define functions $f_1(x,y)$, $f_2(x,y)$ and $f_3(x,y)$ on the set $I^o \times I^o$ in the following way:

$$f_1(x,y) = \begin{cases} \alpha_n^{(o)} + x(y-\alpha_n^{(o)} + 2h_n(y)) & \text{for } x \in (0,\frac{1}{2}] - \{\frac{1}{4}\}, \quad y \in (\alpha_n^{(o)},\beta_n^{(o)}) \\ \alpha_n^{(o)} + \frac{1}{4}(y-\alpha_n^{(o)} + 2\lim\limits_{\eta\to y^+} h_n(\eta)) & \text{for } x = \frac{1}{4}, \quad y \in (\alpha_n^{(o)},\beta_n^{(o)}) \\ y & \text{for } x \in (0,\frac{1}{2}], \quad y \in I^o - G_o \\ \sum\limits_{k=o}^{i-1} \frac{1}{2^k} + \frac{1}{2^i}\alpha_n^{(i)} + \dfrac{(\frac{1}{2})^i}{\sum\limits_{k=1}^{i+1}\frac{1}{2^k}} x(y-\alpha_n^{(i)}) & \text{for } x \in (\sum\limits_{k=1}^{i}\frac{1}{2^k}, \sum\limits_{k=1}^{i+1}\frac{1}{2^k}], \\ & y \in (\alpha_n^{(i)},\beta_n^{(i)}) \\ \sum\limits_{k=o}^{i-1} \frac{1}{2^k} + \frac{1}{2^i} y & \text{for } x \in (\sum\limits_{k=1}^{i}\frac{1}{2^k}, \sum\limits_{k=1}^{i+1}\frac{1}{2^k}], \\ & y \in I^o - G_i \end{cases}$$

$$f_2(x,y) = \begin{cases} y + \sum\limits_{y_i<y} \frac{1}{2^i} & \text{for } x \in I^o, \quad y \in I^o - Z \\ y_n + \sum\limits_{y_i<y_n} \frac{1}{2^i} + x \cdot \frac{1}{2^n} & \text{for } x \in I^o, \quad y_n \in Z \end{cases}$$

$$f_3(x,y) = \quad f_1(x,y) + f_2(x,y) \qquad \text{for } (x,y) \in I^o \times I^o.$$

We shall show that for each $x \in I^o$ the function $(f_3)_x(y)$ is increasing. Let $y' < y''$ and $y' \in I^o - G_o$, $y'' \in G_o$, $x \in (0,\frac{1}{2}]$. Then $(f_1)_x(y) : (\alpha_n^{(o)}, \beta_n^{(o)}) \to (\alpha_n^{(o)}, \beta_n^{(o)})$. There exists $\alpha_n^{(o)}$ such that $y' \leq \alpha_n^{(o)} < y''$. Hence $(f_1)_x(y') < (f_1)_x(y'')$. If

$$x \in (\sum_{k=1}^{i} \frac{1}{2^k} , \sum_{k=1}^{i+1} \frac{1}{2^k}],$$

then $(f_1)_x(y) : (\alpha_n^{(i)}, \beta_n^{(i)}) \to (\sum\limits_{k=o}^{i-1} \frac{1}{2^k} + \frac{1}{2^i} \alpha_n^{(i)}, \sum\limits_{k=o}^{i-1} \frac{1}{2^k} + \frac{1}{2^i} \beta_n^{(i)})$,

$$(f_1)_x(y') = \sum_{k=o}^{i-1} \frac{1}{2^k} + \frac{1}{2^i} y' \leq \sum_{k=o}^{i-1} \frac{1}{2^k} + \frac{1}{2^i} \alpha_n^{(o)} < (f_1)_x(y'').$$

In the other cases our considerations are analogous. The function $(f_1)_x(y)$ is increasing. If $y' < y''$, then

$$(f_2)_x(y') \leq y' + \sum_{y_i \leq y'} \frac{1}{2^i} < y'' + \sum_{y_i<y''} \frac{1}{2^i} \leq (f_2)_x(y'').$$

We obtain that the function $(f_2)_x(y)$ is increasing. Hence $(f_3)_x(y)$ is increasing.

We shall show that for each $y \in A_2$, the function $f_3^y(x)$ is increasing. If $y \in A_2 - Z$, then the function $(f_1)^y(x)$ is increasing because $A_2 - Z \subset H - \tilde{B}$ and

$$(f_1)^y\left(\sum_{k=1}^{i} \frac{1}{2^k}\right) < \lim_{\xi \to \sum_{k=1}^{i} \frac{1}{2^k}+} f_1^y(\xi) .$$

Since the function $(f_2)^y(x)$ is constant, $(f_3)^y(x)$ is increasing. If $y \in Z$, then $(f_1)^y(x)$ is nondecreasing, $(f_2)^y(x)$ is increasing and $(f_3)^y(x)$ is increasing.

Now we show that if $H \cap (I^o - \overline{A}_2) = \emptyset$, then for each $y \in I^o - A_2$, $(f_3)^y(x)$ is not increasing. In this case we have $I^o - A_2 = (I^o - (H \cup Z)) \cup \tilde{B}$. Let $y \in I^o - (H \cup Z)$. Then $(f_1)^y(x)$ is not nondecreasing and $(f_2)^y(x)$ is constant. Hence $(f_3)^y(x)$ is not increasing. Let $y \in \tilde{B}$. Then $(f_1)^y(x)$ is not nondecreasing and $(f_2)^y(x)$ is constant. Hence $(f_3)^y(x)$ is not increasing. In this case the function $f_3(x,y)$ fulfills the required conditions.

If $H \cap (I^o - \overline{A}_2) \neq \emptyset$, we put $G = G_1 \cap (I^o - \overline{A}_2)$ and $G = \bigcup_n (\alpha_n,\beta_n)$, where (α_n,β_n) are components of G,

$$\gamma_n = \lim_{\eta\to\alpha_n^+} f_3(x,\eta), \qquad \delta_n = \lim_{\eta\to\beta_n^-} f_3(x,\eta).$$

For $x = \frac{5}{8}$, $y \in (\alpha_n,\beta_n)$, we change the values of the function $f_3(x,y)$ as in (III). (See the proof of Theorem 1.) In this way we obtain the function $f(x,y)$. We can extend this function to $I \times I$ to obtain a function fulfilling all the conditions.

Necessity. We suppose that $A_1 = I$, $A_2 \subset I$, $\overline{A}_2 - A_2$ does not fulfill condition (*) with respect to A_2, and that there exists a function $f(x,y)$ defined on $I \times I$ such that $A_1 = A_x(f,P)$ and $A_2 = A_y(f,P)$. Let $y \in \overline{A}_2 - A_2$. We have two cases.

Case 1. There exist points $x_0,x_1 \in I$ such that

$$x_0 < x_1 \quad \text{and} \quad f(x_0,y) > f(x_1,y).$$

By Theorem 1 this case may occur only on a finite or countable subset Z of the set $\overline{A}_2 - A_2$. Hence, for each $y \in (\overline{A}_2 - A_2) - Z$, we have

Case 2. There exists an interval $(\alpha_y, \beta_y) \subset I$ such that the function $f^y(x)$ is constant on this interval. Let $B = (\bar{A}_2 - A_2) - Z$. By Lemma 2 there exist a sequence of sets $\{B_n\}_{n\in N}$ and a sequence of intervals $\{P_n\}_{n\in N}$ such that $B = \bigcup_n B_n$ and for every n $\bigcap_{y\in B_n} (\alpha_y, \beta_y) \supset P_n$.

By Remark 2 the set B does not fulfill condition (*) with respect to the set A_2. Therefore, there exists $n_0 \in N$ such that $A_2 \cap B^c_{n_0} > \aleph_0$. Of course $\bigcap_{y\in B_{n_0}} (\alpha_y, \beta_y) \supset P_{n_0}$. Let $x_0, x_1 \in P_{n_0}$ and $x_0 < x_1$. If $y_0 \in A_2 \cap B^c_{n_0}$, then there exists a sequence $\{y_n\}_{n\in N}$ such that $y_n \in B_{n_0}$ and $y_n \to y_0$, but

$$f(x_0, y_0) < f(x_1, y_0) \quad \text{and} \quad f(x_0, y_n) = f(x_1, y_n).$$

Both $f_{x_0}(y)$ and $f_{x_1}(y)$ cannot be simultaneously continuous at the point y_0. So at least one of these functions has an uncountable set of discontinuity points, which is impossible.

Remark 3. If A_2 is a set of the first category on some interval $J \subset I$ and it is c-dense on this interval, then $\bar{A}_2 - A_2$ does not fulfill condition (*) with respect to A_2.

Proof. We assume that $\bar{A}_2 - A_2$ fulfills condition (*) with respect to A_2. Then by the proof of Lemma 1 there exist a set $H \subset J$ of type G_δ and a countable set $Z \subset J$ such that $H = (J \cap A_2 \cup (H \cap (\bar{A}_2 - A_2))) - Z$. Hence $H \cap J$ is a G_δ-set of the first category, dense on the interval J. This contradicts the Baire category Theorem.

Example 1. Let $\{r_i\}_{i\in N}$ be the rational numbers from the interval I^o. For each $n \in N$ let $G_n = \bigcup_{i\in N} (r_i - \frac{1}{2^{i+n+1}}, r_i + \frac{1}{2^{i+n+1}})$, $G = \bigcap_{n\in N} G_n$, and $F = I - G$. Then for $A_1 = I$ and $A_2 = G$ we can construct a function $f(x,y)$ on $I \times I$ such that $A_1 = A_x(f,P)$, $A_2 = A_y(f,P)$. Although $\text{card}(G \cap (\bar{G} - G)^c) > \aleph_0$, we have $\bar{G} - G =$

$\bigcup_{n\in N} (I - G_n)$ and $\mathrm{card}(G \cap (I - G_n)^c) = 0$ for every n, and so, $\bar{G} - G$ fulfills condition (*) with respect to G. Now let $A_1 = I$, $A_2 = F$. The set F is of the first category and c-dense on the interval I. By Remark 3 and Theorem 2 it is not possible to construct a function $f(x,y)$ such that $A_1 = A_X(f,P)$, $A_2 = A_y(f,P)$.

Now let P mean "nondecreasing and continuous".

Theorem 3. There exists a function $f(x,y)$ defined on $I \times I$ such that $A_1 = A_X(f,P)$, $A_2 = A_y(f,P)$ if and only if:

1° $I \neq A_1 \subset I$ and $I \neq A_2 \subset I$ or

2° $A_1 = A_2 = I$ or

3° $A_1 = I$, $A_2 \subset I$ and $I^0 - A_2 = G \cup D$ where G is an open set and D is a subset of the set of one-sided limit points of A_2, or a symmetric condition holds with respect to A_1.

Proof. Sufficiency. If condition 1° or 2° is fulfilled, we define the function $f(x,y)$ by (I) or (II). So suppose that condition 3° holds. Hence $I - A_2 = \bigcup_n P_n$ or $I - A_2$ is the union of $\bigcup_n P_n$ and at least one end-point of the interval $[0,1]$ where P_n is an open interval or a closed interval or a half-open interval. Let the sequences $\{c_n\}_{n\in N}$, $\{d_n\}_{n\in N}$, $\{e_n\}_{n\in N}$ fulfill the following conditions.

$$0 < c_n < c_{n+1}\ ,\quad d_n < d_{n+1}\ ,\quad c_n < e_n < c_{n+1}\ ,$$

$$\lim_{n\to\infty} c_n = 1, \qquad \lim_{n\to\infty} d_n = d < +\infty.$$

We denote by a_n and b_n the end-points of the interval P_n. Let $g_n(y)$ be a linear function for $y \in (a_n,b_n)$, joining the points (a_n,d_n) and (b_n,d_{n+1}). Let $\check{g}_n(y)$ and $\hat{g}_n(y)$ be any continuous functions increasing on (a_n,b_n) such that

$$g_n(y) = \check{g}_n(y) = \hat{g}_n(y) = \begin{cases} d_n & \text{for } y \in [0,a_n] \\ d_{n+1} & \text{for } y \in [b_n,1] \end{cases}$$

$$\check{g}_n(y) < g_n(y) < \hat{g}_n(y) \quad \text{whenever} \quad y \in (a_n,b_n).$$

Now we construct the function $f(x,y)$. In all cases we let $f(x,y) = d$ for $(x,y) \in [0,c_1) \times [0,1]$ and for each n put $f(e_n,y) = g_n(y)$ for $y \in [0,1]$. First we consider the case $P_n = [a_n,b_n]$. Then at the remaining points of the closed trapezoid with vertices $(c_n,0)$, $(e_n,0)$, (e_n,b_n), $(c_n,1)$ we put $f(x,y) = d_n$. At the remaining points of the closed trapezoid with vertices (e_n,a_n), $(c_{n+1},0)$, $(c_{n+1},1)$, $(e_n,1)$ we put $f(x,y) = d_{n+1}$. On the triangle with vertices $(c_n,1)$, (e_n,b_n), $(e_n,1)$ we define the function $f(x,y)$ in such a way that all sections $f^y(x)$ for $y \in (b_n,1)$ are linear functions joining the points $\left(e_n - \dfrac{(e_n-c_n)(y-b_n)}{1-b_n}, d_n\right)$ and (e_n,d_{n+1}) for $x \in \left(e_n - \dfrac{(e_n-c_n)(y-b_n)}{1-b_n}, e_n\right)$. In a similar way we define $f(x,y)$ on the triangle completing the rectangle bounded by $x = c_n$ and $x = c_{n+1}$.

We now consider the case $P_n = (a_n,b_n)$. We define the function $f(x,y)$ so that all sections $f^y(x)$ for $y \in [0,1]$ are linear functions joining the points (c_n,d_n) and $(e_n,\check{g}_n(y))$ for $x \in [c_n,e_n)$ and the points $(e_n,\hat{g}_n(y))$ and (c_{n+1},d_{n+1}) for $x \in (e_n,c_{n+1}]$. If $P_n = (a_n,b_n]$, then on $[c_n,e_n) \times [0,1]$ we construct $f(x,y)$ as in the first case and on $(e_n,c_{n+1}] \times [0,1]$ we construct $f(x,y)$ as in the second case. If $P_n = [a_n,b_n)$, we proceed symmetrically. For $y \in [0,1]$ we put $f(1,y) = d$. If $0 \notin A_2$ or $1 \notin A_2$, it is not difficult to make a modification of this definition so that the function $f(x,y)$ will satisfy all the required conditions.

Necessity. We suppose that conditions 1°, 2°, 3° are not fulfilled. Then $A_1 = I$ and there exists a point y_0 in $I^0 - A_2$ which is a bilateral limit point of A_2. We assume that there exists a function $f(x,y)$ such that $A_1 = A_x(f,P)$ and $A_2 = A_y(f,P)$. Then the function $f^{y_0}(x)$ is not nondecreasing or is not continuous. In the first case from the proof of Theorem 1 it follows that there exists an interval P_0 such that for each $x \in P_0$ the function $f_x(y)$ is not continuous. We obtain a contradiction. In the second case we assume that the function $f^{y_0}(x)$ is not continuous at a point x_0. Then we have

$$(2) \qquad f^{y_0}(x_0) < \lim_{\xi\to x_0^+} f^{y_0}(\xi) = a \qquad \text{or}$$

(3) $$b = \lim_{\xi \to x_0^-} f^{y_0}(\xi) < f^{y_0}(x_0).$$

From (2) it follows that for each $x \in I$ if $x > x_0$, then $f_x(y_0) \geq a$. There exists $y_1 \in A_1$, $y_1 > y_0$, such that $f(x_0,y_1) < a$. But $\lim_{\xi \to x_0^+} f\ (\xi) = f(x_0,y_1)$. So there exists $x_1 > x_0$ such that $f(x_1,y_1) < a$ which is a contradiction.

From (3) in an analogous way we obtain a contradiction.

Remark 4. If $A_1 = I$, $I \neq A_2 \subset I$ and if there exists a function $f(x,y)$ such that $A_1 = A_x(f,P)$, $A_2 = A_y(f,P)$, then A_2 is a set of type G_δ.

Definition 2. We say that a set D fulfills condition (**) with respect to A_2 if and only if there exists a sequence set $\{D_n\}_{n \in N}$ such that $D = \bigcup_n D_n$ and for every n $A_2 \cap D_n^c = \emptyset$.

Remark 5. If D does not fulfill condition (**) with respect to A_2, then the set $D - Z$ where Z is any countable set does not fulfill this condition.

Remark 6. Let a function $g(x,y)$ be defined on $[\alpha,\beta] \times [0,1]$ where $\beta - \alpha < 1$ such that $g_x(y)$ are increasing functions for each $x \in [\alpha,\beta]$ and $g^y(x)$ are nondecreasing functions for each $y \in [0,1]$. Then there exists a function $g_1(x,y)$ defined on $[\alpha,\beta] \times [0,1]$ such that $g_1(x,y)$ is increasing on every vertical and horizontal section of the triangle with vertices $(\alpha,0)$, $(\beta,0)$, $(\beta,\beta - \alpha)$ and $g_1(x,y) = g(x,y)$ on the complement of this triangle in $[\alpha,\beta] \times [0,1]$.

A similar result can be obtained with respect to the triangle with vertices $(\alpha,1 + \alpha - \beta)$, $(\beta,1)$, $(\alpha,1)$.

Proof. We project orthogonally all points of this triangle on its hypotenuse. Let the value of g_1 at (x,y) be equal to the value of g at the projection of (x,y). At the remaining points of the rectangle we do not change the function g. It is easy to verify that the function g_1 has the required properties.

Let P now mean "increasing and continuous".

Theorem 4. There exists a function $f(x,y)$ on the set $I \times I$ such that $A_1 = A_x(f,P)$, $A_2 = A_y(f,P)$ if and only if:

1° $I \neq A_1 \subset I$ and $I \neq A_2 \subset I$ or

2° $A_1 = A_2 = I$ or

3° $A_1 = I$, $A_2 \subset I$ and $\bar{A}_2 - A_2$ fulfills condition (**) with respect to the set A_2 or

$A_2 = I$, $A_1 \subset I$, $\bar{A}_1 - A_1$ fulfills condition (**) with respect to the set A_1.

Proof. Sufficiency. Use Theorem 1 if 1° or 2° occur. Let $B = \bar{A}_2 - A_2$ and suppose that B fulfills condition (**) with respect to A_2. Then $B = \bigcup_n B_n$ and for every n $A_2 \cap B_n^c = \emptyset$. Accordingly there exist open sets G_n such that $I^o \cap A_2 \subset G_n$ and $\operatorname{card}(B_n \cap G_n) \leq \aleph_0$. Let $H = \bigcap_n G_n$. Then $I^o \cap A_2 \subset H$ and $\operatorname{card}(B \cap H) \leq \aleph_0$. We can assume that $G_{n+1} \subset G_n$ for $n \in N$. Let $\{c_n\}_{n\in N}$ and $\{d_n\}_{n\in N}$ be sequences such that $0 < c_n < c_{n+1}$ for $n \in N$, $\lim_{n\to\infty} c_n = c < 1$, $\sum_{n=1}^{\infty} \gamma_n < +\infty$ where $\gamma_n = c - c_n$ and $c < d_1$, $d_n < d_{n+1}$ for $n \in N$, $\lim_{n\to\infty} d_n = 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$ where $\delta_n = 1 - d_n$. If $H \cap (I^o - \bar{A}_2) = \emptyset$, then we put $f_1(x,y) = x \cdot y$ for $(x,y) \in [0,\frac{c_1}{2}] \times [0,1]$. If $H \cap (I^o - \bar{A}_2) \neq \emptyset$, then $G_1 \cap (I^o - \bar{A}_2) = \bigcup_i (a_i,b_i)$ and we change the function $f_1(x,y)$ for $x = \frac{c_1}{4}$, $y \in (a_i,b_i)$ similarly as in (III). (See the proof of Theorem 1.) We denote the elements of the set $H \cap B$ by $\{\alpha_n\}$. This set is finite or countable. We define a sequence $\{\beta_n\}_{n\in N}$ such that

$$\beta_1 - \alpha_1 \gamma_1 = 1,$$

$$\beta_{n+1} = \beta_n + \gamma_{2n-1}(1 - \alpha_n)c_{2n} + \alpha_{n+1} \cdot \gamma_{2n+1}$$

and

$$\lim_{n\to\infty} \beta_n = \beta.$$

This is possible because $\sum_{n=i}^{\infty} \gamma_n < \infty$ and $c_n < 1$ for every n. Now we define the function $f_1(x,y)$ on the rectangles $[c_{2n-1}, c_{2n}] \times [0,1]$ by

$$f_1(x,y) = \begin{cases} \beta_n + (y-\alpha_n)(c-x) & \text{for } x \in [c_{2n-1},c_{2n}],\ y \in [0,\alpha_n] \\ \beta_n & \text{for } x \in [c_{2n-1},c_{2n}],\ y = \alpha_n \\ \beta_n + \gamma_{2n-1} \cdot x \cdot (y-\alpha_n) & \text{for } x \in [c_{2n-1},c_{2n}],\ y \in (\alpha_n,1]. \end{cases}$$

Let $f_1(c,y) = \beta$ for $y \in [0,1]$. We have $G_n = \bigcup_i (a_i^{(n)}, b_i^{(n)})$ for $n \in \mathbb{N}$.

Next we define a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ such that $\eta_1 > \beta$, $\eta_{n+1} = \eta_n + \delta_{n-1}$ and for every n a sequence $\{\varepsilon_i^{(n)}\}_{i\in\mathbb{N}}$ such that $0 < \varepsilon_i^{(n)} < \dfrac{b_i^{(n)} - a_i^{(n)}}{2}$.

We construct the function $f_1(x,y)$ on $[d_{2n-1}, d_{2n}] \times [0,1]$ by

$$f_1(x,y) = \begin{cases} \eta_n + \delta_{2n-1} a_i^{(n)} + \delta_{2n-1} \cdot x(y - a_i^{(n)}) & \text{for } x \in [d_{2n-1},d_{2n}], \\ & y \in (a_i^{(n)}, a_i^{(n)}+\varepsilon_i^{(n)}] \\ \eta_n + \delta_{2n-1} b_i^{(n)} + (1-x)\delta_{2n-1}(y-b_i^{(n)}) & \text{for } x \in [d_{2n-1},d_2], \\ & y \in [b_i^{(n)}-\varepsilon_i^{(n)}, b_i^{(n)}) \\ \text{a linear function joining the points } & (a_i^{(n)}+\varepsilon_i^{(n)}, f_1(x,a_i^{(n)}+\varepsilon_i^{(n)})) \\ \text{and } (b_i^{(n)}-\varepsilon_i^{(n)}, f_1(x,b_i^{(n)}-\varepsilon_i^{(n)})) & \text{for } x \in [d_{2n-1},d_{2n}] \\ & y \in (a_i^{(n)}+\varepsilon_i^{(n)}, b_i^{(n)}-\varepsilon_i^{(n)})) \\ \eta_n + \delta_{2n-1} \cdot y & \text{for } x \in [d_{2n-1},d_{2n}] \\ & \text{and for the remaining } y \in [0,1]. \end{cases}$$

Let $f_1(1,y) = \lim_{n\to\infty} \eta_n < +\infty$. The function $f_1(x,y)$ on the intervals $[\frac{c_1}{2}, c_1]$, $[c_{2n}, c_{2n+1}]$, $[c, d_1]$, $[d_{2n}, d_{2n+1}]$ is for each $y \in [0,1]$ a linear function joining the value of the function f_1^y at the left end-point of the above-mentioned intervals and the value of the function f_1^y at the right end-point of these intervals. If $\{0\} \cup \{1\} \subset I - A_2$, then one may verify as in the proof of Theorem 2 that the function $f_1(x,y) + y$ satisfies all the required conditions. If $0 \in A_2$, we use Remark 6 to construct a function $f_1(x,y)$ such that $f_1(x,y) + y$ satisfies all the required conditions.

We proceed similarly in the case when $1 \in A_2$.

Necessity. We assume that $A_1 = I$, $A_2 \subset I$ and the set $B = \bar{A}_2 - A_2$ does not fulfill condition (**) with respect to the set A_2. Let Z be the set of one-sided limit points of the set A_2. Let $y_0 \in B - Z$. If the function $f^{y_0}(x)$ is not continuous, then by the proof of Theorem 3 there exists x_0 such that the function $f_{x_0}(y)$ is not continuous. This contradicts the equality $A_X(f,P) = I$. If $f^{y_0}(x)$ is not increasing, then there exist points x_0, $x_1 \in I$ such that $x_0 < x_1$ and $f(x_0,y_0) > f(x_1,y_0)$ or $f^{y_0}(x)$ is constant on some interval $(\alpha_{y_0},\beta_{y_0})$. The first case by the proof of Theorem 1 is not possible. So the function $f^y(x)$ is constant on the interval (α,β_y) where $y \in B - Z$. By Lemma 2 there exist a sequence of sets $\{B_n\}_{n \in N}$ and a sequence of intervals $\{P_n\}_{n \in N}$ such that

$$B - Z = \bigcup_n B_n \quad \text{and} \quad \bigcap_{y \in B_n} (\alpha_y,\beta_y) \supset P_n \quad \text{for every } n.$$

By Remark 5 it follows that the set $B - Z$ does not satisfy condition (**) with respect to the set A_2. Therefore, there exists $n_0 \in N$ such that $A_2 \cap B_{n_0}^c \neq \emptyset$ and by Lemma 2

$$\bigcap_{y \in B_{n_0}} (\alpha_y,\beta_y) \supset P_{n_0}.$$

We obtain a contradiction just as in the proof of Theorem 2.

We obtain the following as we did Remark 3.

Remark 7. If a set A_2 is a set of the first category in some interval $J \subset I$ and if it is dense in this interval, then the set $\bar{A}_2 - A_2$ does not fulfill condition (**) with respect to A_2.

Corollary 1. Let $A_1 = I$ and let A_2 be the set of all rational numbers from the interval I. Then there does not exist a function such that $A_1 = A_X(f,P)$, $A_2 = A_y(f,P)$.

Remark 8. The set G defined in Example 1 fulfills condition (**) while $I - G$ does not.

Remark 9. If $A_1 = I$, $I \neq A_2 \subset I$ and there exists a function $f(x,y)$ such that $A_1 = A_X(f,P)$, $A_2 = A_y(f,P)$, then the set A_2 must be both a $G_{\delta\sigma}$-set and an $F_{\sigma\delta}$-set.

Lastly, let P mean "of bounded variation".

Theorem 5. For any sets $A_1 \subset I$, $A_2 \subset I$, there exists a function $f(x,y)$ such that $A_1 = A_X(f,P)$, $A_2 = A_y(f,P)$.

Proof. We assume that $I - A_1 \neq \emptyset$ and $I - A_2 \neq \emptyset$ and for each $y \in I - A_2$ we choose a sequence $B_y = \{x_y^{(n)}\}_{n\in\mathbf{N}}$ such that the sets B_y are mutually disjoint and $\bigcup_{y\in I-A_2} B_y \subset I$ (a subset of the x-axis). For each $x \in I - A_1$ we choose a sequence $A_X = \{y_x^{(n)}\}_{n\in\mathbf{N}}$ such that the sets A_X are mutually disjoint and $\bigcup_{x\in I-A_1} A_X \subset I$ (a subset of the y-axis).

We define

$$f_1(x,y) = \begin{cases} \frac{1}{n} & \text{for } x = x_y^{(n)}, \quad y \in I - A_2 \\ 0 & \text{for the remaining } (x,y) \in I \times I \end{cases}$$

and

$$f(x,y) = \begin{cases} \frac{1}{n} & \text{for } x \in I - A_1, \quad y = y_x^{(n)} \\ f_1(x,y) & \text{for the remaining } (x,y) \in I \times I. \end{cases}$$

It is easy to verify that $f(x,y)$ satisfies all the required conditions. If $I - A_1 = \emptyset$, then $f_1(x,y)$ is the required function. The case $I - A_2 = \emptyset$ is symmetric and the case $I - A_2 = \emptyset$, $I - A_1 = \emptyset$ is obvious.

Received June 26, 1985