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Measurability of Real Functions Having Symmetric Derivatives Everywhere

In 1928, Sierpinski posed the question whether there is a nonmeasurable function f whose symmetric derivative $f^s(x) = 0$ at every real number x [5]. Preiss gave a negative answer [4]. In fact, his result shows that a real function f having finite $f^s(x)$ at every x is continuous almost everywhere and hence measurable. This leads to a stronger form of this type of question [3]: Is f measurable if $f^s(x)$ exists (finite or not) at every x? An affirmative answer is contained in a general theorem proved by Uher [7]. Here, based on recent work done by Belna, Evans, Humke, Larson, and Thomson (see [1], [2], [3], and [6]), the authors give a new proof for the following.

Theorem. If a function f has a symmetric derivative $f^s(x)$ at every x, then f is measurable.

Throughout this paper, let f be a real function for which $f^s(x)$ exists at every x, C the set of points where f is continuous, C^s the set of points where f is symmetrically continuous and D^s the set of points where f^s is finite. For a set E of real numbers, $|E|_i$ denotes the interior Lebesgue measure of E while $|E|$ denotes the Lebesgue measure of E if E is measurable. Also, $\bar{E}$ and E^o denote the closure and the interior of E respectively. It should be noted that the sets D^s, $\{x : f^s(x) = +\infty\}$ and $\{x : f^s(x) = -\infty\}$ are measurable since f^s is in the first Baire class [3].

Lemma. $|\{x : |f^s(x)| = \infty\} \cap I| < |I|$ for every interval I.

Proof. Let $A = \{x : |f^s(x)| = \infty\}$. For a given interval I, it is trivial that $|A \cap I| < |I|$ if A is not dense in I. We assume that A is dense in I. Let A_+ and A_- denote the sets $\{x : f^s(x) = +\infty\}$ and $\{x : f^s(x) = -\infty\}$ respectively. If $(\bar{A}_+)^o \cap I \neq \emptyset$, then there exists an interval $I_1 \subset I$ such that A_+ is dense in I_1. Otherwise, A_- is dense in I. We prove for the first case only. (The second case is proved by considering $-f$.) Now, since f^s is in the first Baire class, A_+ is dense in I_1 implies that $\{x: f^s(x) \leq 0\}$ is not dense in I_1. There exists an open interval $J \subset I_1$ such that $J \subset \{x : f^s(x) > 0\}$. Thus A_+ is dense in J and $A \cap J = A_+ \cap J$. Clearly it is sufficient to show that $|A_+ \cap J| < |J|$.

If $|A_+ \cap J| = |J|$, then $|D^s \cap J| = 0$. There exists an open set G with $D^s \cap J \subset G \subset J$ and $|G| < |J|/4$. For each $x \in D^s \cap J$, since $f^s(x) > 0$ and $x \in G$, there is a $\delta_x > 0$ such that

$$f(x+h) - f(x-h) > 0 \quad \text{whenever } 0 < h < \delta_x$$

and

$$(x-\delta_x , x+\delta_x) \subset G.$$

For each positive integer n and each $x \in A_+ \cap J$, since $f^s(x) > n$, there is a $\delta(x,n) > 0$ such that

$$f(x+h) - f(x-h) > 2hn \quad \text{whenever } 0 < h < \delta(x,n).$$

Let $\mathcal{J}'' = \{[x-h , x+h] : x \in D^s \cap J,\ 0 < h < \delta_x\}$. For each n, let $\mathcal{J}'_n = \{[x-h , x+h] : x \in A_+ \cap J,\ 0 < h < \delta(x,n)\}$ and $\mathcal{J}_n = \mathcal{J}'_n \cup \mathcal{J}''$. Then, for each n, $\mathcal{J}_n$ is a symmetric full cover of J according to Thomson [6] and, by his Lemma 3.1, there exists $S_n \subset J_r$ such that $\overline{J_r - S_n}$ is countable and $\mathcal{J}_n$ contains a partition of $[c-x , c+x]$ for every x with $c+x \in S_n$, where J_r is the right half of J and c is the midpoint of J.

Let $S = \bigcap S_n$ and J_{rr} be the right half of J_r. Clearly $J_r - S$ is countable and $J_{rr} \cap S \neq \emptyset$. If $b \in J_{rr} \cap S$, then $b - c > |J_r|/2 = |J|/4$. For each n, $\mathcal{J}_n$ contains a partition of $[a , b]$, say $J_1^n, J_2^n, \cdots, J_{k_n}^n$, where $a = 2c - b$. Let $f([a , b]) = f(b) - f(a)$. Then

$$f([a , b]) = \sum\{f(J_k^n) : k = 1, \cdots, k_n\}$$

$$= \sum\{f(J_k^n) : J_k^n \in \mathcal{J}_n'\} + \sum\{f(J_k^n) : J_k^n \in \mathcal{J}''\}.$$

Noting that $J_k^n \in \mathcal{J}''$ implies $J_k^n \subset G$ and $|G| < |J|/4 < b - c = (b-a)/2$, we see that there must be a $k \in \{1, \cdots, k_n\}$ such that $J_k^n \in \mathcal{J}_n'$. Since $f(J_k^n) > 0$ for $J_k^n \in \mathcal{J}''$ and $f(J_k^n) > n|J_k^n|$ for $J_k^n \in \mathcal{J}_n'$, we have, for every n,

$$f([a, b]) > n \sum\{|J_k^n| : J_k^n \in \mathcal{J}_n'\}$$

$$= n\,|[a, b] - \cup\{J_k^n : J_k^n \in \mathcal{J}''\}|$$

$$> n(|[a, b]| - |G|)$$

$$> n(\tfrac{1}{2}|J| - \tfrac{1}{4}|J|) = \tfrac{n}{4}|J| .$$

This is a contradiction to the fact that $f([a, b])$ is finite. The lemma is proved.

Proposition. C is the complement of a σ-porous set.

Proof. Firstly we show that C is dense. Let an interval I be given. By the lemma, $|D^s \cap I| > 0$. Belna proved [1] that $|C^s - C|_i = 0$. Noting that $D^s \cap I - C$ is a measurable subset of $C^s - C$, we have $|D^s \cap I - C| = 0$ and hence $|D^s \cap I \cap C| > 0$, $I \cap C \neq \emptyset$.

By a theorem of Belna, Evans and Humke [2], $f'(x)$ exists at every x except on a σ-porous set. Thus it suffices to show that the sets $B_+ = \{x : f'(x) = +\infty,\ x \notin C\}$ and $B_- = \{x : f'(x) = -\infty,\ x \notin C\}$ are σ-porous. In fact, they are countable. For x with $f'(x) = +\infty$, we have

$$\varliminf_{t \to x-} f(t) \leq \varlimsup_{t \to x-} f(t) \leq f(x) \leq \varliminf_{t \to x+} f(t) \leq \varlimsup_{t \to x+} f(t).$$

It follows that

$$B_+ \subset \{x : \varliminf_{t \to x-} f(t) < \varliminf_{t \to x+} f(t)\} \cup \{x : \varlimsup_{t \to x-} f(t) < \varlimsup_{t \to x+} f(t)\}$$

and B_+ is countable. Analogously, B_- is countable. The proof is completed.

Since a σ-porous set is of measure zero, the Theorem follows immediately.

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