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Solution of a Problem Concerning Functions of Harmonic Bounded Variation

It is known [3] that the space of all regulated functions for which the Fourier series converges for every change of variable (GW) contains the space of functions of harmonic bounded variation (HBV) and the space of all functions for which the Fourier series converges uniformly for every change of variable (UGW) contains the space of continuous functions of harmonic bounded variation (HBV_c). In [3, p. 17] and [4] the question is raised whether $GW = HBV$ and $UGW = HBV_c$. In [5] it is pointed out that [1] contains the result $HBV_c \subseteqq UGW \subsetneqq GW_c$, implying $HBV \subsetneqq GW$. The purpose of the present note is to prove that $HBV_c \subsetneqq UGW$. I would like to express my thanks to RN Dr. L. Zajíček for acquainting me with the problem.

The function $f : R \to R$ is said to be regulated if it has right and left limits at each point. The space of all regulated functions of period 2π will be denoted by $P(2\pi)$. If X is a class of functions, then X_c will denote the continuous functions in class X.

The finite system of nonoverlapping intervals $\{I_i\}$ $i = 1,\dots,n$ is said to be an ordered system if I_i is to the left of I_{i+1} for all $i = 1,\dots,n-1$, or if I_i is to the right of I_{i+1} for every $i = 1,\dots,n-1$.

If $I = [a,b] \subset [-\pi,\pi]$ and $f \in P(2\pi)$, then we write $f(I) = f(b) - f(a)$. For $f \in P(2\pi)$ we will say that $f \in HBV$ if

$$\sup \sum_{i=1}^{n} |f(I_i)|/i = V_H(f) < \infty,$$

where the supremum is taken over all finite systems of nonoverlapping, closed intervals in $[-\pi,\pi]$. For $f \in P(2\pi)$ we will say that $f \in OHBV$ if

$$\sup \sum_{i=1}^{n} |f(I_i)| / i = V_{OH}(f) < \infty,$$

where the supremum is taken over all finite ordered systems of nonoverlapping, closed intervals in $[-\pi,\pi]$. For $f \in P(2\pi)$ we will say that $f \in GW$ if the Fourier series of $f \circ g$ converges everywhere for every homeomorphism g of $[-\pi,\pi]$ with itself, and $f \in UGW$ if the Fourier series of $f \circ g$ converges uniformly for every homeomorphism g of $[-\pi,\pi]$ with itself.

<u>Theorem:</u> $UGW \supsetneqq HBV_c$.

Proof: There exists [2] a function $f \in OHBV - HBV$. Let n be a nonnegative integer. Since $f \notin HBV$, there exists a system of nonoverlapping, closed intervals $\{I_i^n\}$, $i = 1,\ldots,m_n$ in $[-\pi,\pi]$ such that

$$\sum_{i=1}^{m_n} |f(I_i^n)| / i > 2^n\, 10^n.$$

Let $V_{OH}(f) = c$ and let $I_i^n = [a_i^n,b_i^n]$, $i = 1,\ldots,m_n$. Define $h_n(x) = 0$ for $x \in [-\pi,\pi] - \bigcup_{i=1}^{m_n} (a_i^n,b_i^n)$, $h_n((a_i^n + b_i^n)/2) = |f(I_i^n)|$ for $i = 1,\ldots,m_n$ and extend h_n linearly to the remainder of $[-\pi,\pi]$. Clearly $|h_n(x)| \leq c$ for all $x \in [-\pi,\pi]$, $V_H(h_n) \geq 2^n\, 10^n$ and $V_{OH}(h_n) \leq 2c$. Let p_n be the increasing, linear mapping of $[2^{-n},2^{-(n-1)}]$ onto $[-\pi,\pi]$. Define the function $w_n \in P(2\pi)_c$ by setting $w_n(y) = 2^{-n}\, h_n(p_n(y))$ for $y \in [2^{-n},2^{-(n-1)}]$ and $w_n(y) = 0$ for $y \in [-\pi,2^{-n}) \cup (2^{-(n-1)},\pi]$. It is easy to see that w_n is continuous, $V_{OH}(w_n) \leq 2c / 2^n$ and $|w_n(x)| \leq c2^{-n}$ for all $x \in R$. Put $H = \sum_{i=1}^{\infty} w_i$. Evidently $H \in P(2\pi)_c$. Let $J_i^n = p_n^{-1}([a_i^n,(a_i^n + b_i^n)/2])$.

Then $\{J_i^n\}$ is a system of nonoverlapping, closed intervals in $[-\pi,\pi]$ and

$$\sum_{i=1}^{m_n} |H(J_i^n)|/i = \sum_{i=1}^{m_n} |w_n(J_i^n)|/i = \sum_{i=1}^{m_n} 2^{-n} h_n([a_i^n,(a_i^n+b_i^n)/2])/i =$$

$$= 2^{-n} \sum_{i=1}^{m_n} |f(I_i^n)|/i \geq 10^n.$$ Consequently $H \notin HBV_c$. A necessary and sufficient condition for a function H to be in UGW is the following condition. (See [3], p. 15 or [1].)

(P) For every $\varepsilon > 0$ there is $\delta > 0$ such that for every ordered system of nonoverlapping, closed intervals $\{I_i\}$, $i = 1,\dots,k$ in $[-\pi,\pi]$ for which $\operatorname{diam}(\bigcup_{i=1}^{k} I_i) < \delta$ the inequality $\sum_{i=1}^{k} |H(I_i)|/i < \varepsilon$ holds.

Let $\varepsilon > 0$ be given. Then choose a positive integer n for with $c/2^{n-2} < \varepsilon$. Put $f_1 = \sum_{i=1}^{n} w_i$ and $f_2 = \sum_{i=n+1}^{\infty} w_i$. Clearly f_1 is a Lipschitz function. Let L be its Lipschitz constant. Put $\delta = \varepsilon/2L$. Let $\{I_i\}$, $i = 1,\dots,k$ be an ordered system of closed, non-overlapping intervals in $[-\pi,\pi]$ with $\operatorname{diam}(\bigcup_{i=1}^{k} I_i) < \delta$. Then

$$\sum_{i=1}^{k} |H(I_i)|/i \leq \sum_{i=1}^{k} |f_1(I_i)|/i + \sum_{i=1}^{k} |f_2(I_i)|/i \leq \sum_{i=1}^{k} L \operatorname{diam}(I_i)/i +$$

$$+ \sum_{i=1}^{k} \Big|\Big(\sum_{j=n+1}^{\infty} w_j\Big)(I_i)\Big|/i \leq L\delta + \sum_{i=1}^{k} \sum_{j=n+1}^{\infty} |w_j(I_i)|/i \leq$$

$$\leq \varepsilon/2 + \sum_{j=n+1}^{\infty} \sum_{i=1}^{k} |w_j(I_i)|/i \leq \varepsilon/2 + \sum_{j=n+1}^{\infty} V_{OH}(w_j) \leq$$

$$\leq \varepsilon/2 + \sum_{j=n+1}^{\infty} 2c/2^j \leq \varepsilon/2 + c/2^{n-1} < \varepsilon.$$

Hence the condition (P) is satisfied and consequently $H \in UGW$. The Function H is in UGW but not in HBV_c.

REFERENCES

[1] A. Baernstein and D. Waterman, Function whose Fourier series converge uniformly for every change of variable, Indiana Univ. Math. J., Vol. 22, No. 6, 1972, pp. 569-576.

[2] C.L. Belna, On ordered harmonic bounded variation, Proc. Amer. Math. Soc., Vol. 80, No. 3, 1980, pp. 441-444.

[3] C. Goffman, Lebesque equivalence, Real Analysis Exchange, Vol. 3, No.1, 1977-78, pp. 7-23.

[4] D. Waterman, On the note of C.L. Belna, Proc. Amer. Math. Soc., Vol. 80, No. 3, 1980, pp. 445-447.

[5] ___________, New results on function classes invariant under change of variable, Real Analysis Exchange, Vol. 9, No. 1, 1983-84, pp. 146-153.

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