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Measures for Which σ-Porous Sets are Null

A set $E \subseteq R$ is said to be porous at a point x if

$$p(E,x) = \limsup \rho(x,h)/h > 0$$

where $\rho(x,h)$ is the length of the longest subinterval of $(x-h,x+h) \cap E^c$. The set E is a porous set if E is porous at each of its points, and E is σ-porous if it is the denumerable union of porous sets. Although it is evident that every σ-porous set is both of measure zero and of the first Baire category, the reverse implication is not true ([HT] or [Z]), and in applications it is often the geometry of porous sets which carries the important information. In recent years, the notion of porosity has played an important role in characterizing sets of exceptional derivative or derivate behavior and in many instances has been used to improve older results which use sets of Lebesgue measure zero which are also of the first Baire category. An interesting survey of these results can be found in B.S. Thomsons work, [TH]. One advantage to using measure is that a great deal can be brought to bear on a problem in the context of measure theory. It is natural, then, to try to define a nontrivial Borel measure on a given set in such a way that the σ-porous subsets are necessarily null. This would enable one to use the associated measure theory to study exceptional behavior. In [Tk], J. Tkadlec considered this problem and constructed a certain perfect set E of Lebesgue measure zero such that if μ is any nontrivial Borel measure on E, then there are porous subsets of E which have positive μ-measure. We call a Borel measure μ on a set E a <u>Tkadlec</u> <u>measure</u> if there is a porous subset of E which has positive μ-measure. If a set E is itself σ-porous then it is evident that every nontrivial Borel measure on E is a Tkadlec measure. On the other hand, if a set E is of positive Lebesgue measure, then Lebesgue measure itself is a non-Tkadlec measure on E. In this paper we determine exactly which symmetric perfect sets have the property that every nontrivial Borel measure is a Tkadlec measure.

If $\{a_n\}_{n=1}^{\infty}$ is a given sequence with $0 < a_n < 1$ for n=0,1,2,..., then the symmetric Cantor set $C(a_n)$ determined by the sequence $\{a_n\}_{n=1}^{\infty}$ is defined as follows. First, an interval of length a_1 is centrally deleted from [0,1] leaving two disjoint intervals, $J_{1,1}$ and $J_{1,2}$, each of length say L_1. Then an interval of length a_2L_1 is centrally deleted from each of $J_{1,1}$ and $J_{1,2}$ leaving four interval, $J_{2,i}$ i=1,2,3,4, and so on. The symmetric Cantor set $C(a_n)$ is then defined as

$$C(a_n) = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} J_{n,i}$$

It is evident that $C(a_n)$ is of positive Lebesgue measure if and only if $\sum_{n=1}^{\infty} a_n < \infty$. It is not quite so evident that $C(a_n)$ is non σ-porous if and only if $\{a_n\}_{n=1}^{\infty} \to 0$, [HT]. Our characterization is also given in terms of the defining sequence, $\{a_n\}_{n=1}^{\infty}$, and is:

THEOREM. <u>Every nontrivial Borel measure on the symmetric Cantor set</u> $C(a_n)$ <u>is a Tkadlec measure if and only if</u> $\sum_{n=1}^{\infty} a_n^s = \infty$ <u>for every</u> s.

In particular, if $s \leq 1$ and $\sum_{n=1}^{\infty} \alpha_n^s = \infty$, then Lebesgue measure is a non-Tkadlec measure on $C(\alpha_n)$. If $\alpha_n = \frac{1}{\ln(n+2)}$ then it follows from our theorem that every nontrivial Borel measure on $C(\alpha_n)$ is a Tkadlec measure. If, for a third example, $\alpha_n = \frac{1}{n+2}$, then $C(\alpha_n)$ is of Lebesgue measure zero and is not σ-porous. However, our theorem yields a nontrivial Boral measure μ on $C(\alpha_n)$ such that every porous subset of $C(\alpha_n)$ is of μ-measure zero.

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