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ON THE BOUNDARY VALUE OF BESOV-BERGMAN SPACES

Dedicated to Arnaldo Antonio de Souza and Corina Soares de Souza, my parents.

Let b denote a special atom, $b\colon [-\pi,\pi) \to R$, $b(t) = 1/2\pi$ or for any interval I in $[-\pi,\pi)$, $b(t) = -|I|^{-1/p}\chi_R(t) + |I|^{-1/p}\chi_L(t)$, L is the left half of I, R the right half, $|I|$ denotes the length of I and χ_E the characteristic function of E. For $1/2 < p < \infty$, let (b_n) be special atoms and (c_n) a sequence of real numbers, then we define the space $B^p = \{f\colon [-\pi,\pi) \to R;\ f(t) = \sum c_n b_n(t),\ \sum |c_n| < \infty\}$. We endow B^p with norm $\|f\|_{B^p} = \operatorname{Inf} \sum|c_n|$, where the infimum is taken over all possible representations of f.

These spaces were originally introduced by the author who has extensively studied them. The reader is referred to [2], [3], [4], [5], [6], [7], [8], [9] and [10].

In the early 1960's the following spaces were introduced, now known as Besov-Bergman Spaces. For $0 < \alpha < 1$, $1 \le r, s \le \infty$, let

$$\Lambda(\alpha,r,s)=\{f\colon [-\pi,\pi) \to R;\ \|f\|_{\Lambda(\alpha,r,s)}= \|f\|_r + \left(\int_{-\pi}^{\pi} \frac{(\|f(x+t)-f(x)\|_r)^s}{|t|^{1+\alpha s}}\,dt\right)^{1/s} < \infty\}.$$

where $\|\ \|_r$ is the Lebesgue Space L^r-norm.

These spaces have been studied in depth in [1], [11], [12], [13] and [14].

We have the following result.

THEOREM A (Embedding Theorem). If $f \in B^p$, $1 < p < \infty$, then $f \in \Lambda(1 - \frac{1}{p}, 1,1)$. Moreover $\|f\|_{\Lambda(1-\frac{1}{p}, 1,1)} < M\|f\|_{B^p}$, where M is an absolute constant.

Because of Theorem A we can regard B^p as a subset of $\Lambda(1-1/p,1,1)$, so that we have:

LEMMA B B^p is a dense subset of $\Lambda(1 - \frac{1}{p}, 1,1)$.

Theorem A and Lemma B, in addition to the fact that the dual space of B^p coincides with that of $\Lambda(1 - \frac{1}{p}, 1,1)$, see [6], [8] implies the following result:

THEOREM C $f \in B^p$ for $1 < p < \infty$ if and only if $f \in \Lambda(1 - \frac{1}{p}, 1,1)$. Moreover there are absolute constants M and N such that

$$M\|f\|_{B^p} \leq \|f\|_{\Lambda(1 - \frac{1}{p}, 1,1)} \leq N\|f\|_{B^p}.$$

We recall that one of our earlier results with Richard O'Neil and G. Sampson, see [8], was that B^p is equivalent with the spaces of all analytic functions in the disk for which

$$\frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} \left|F'(re^{i\theta})\right| (1-r)^{\frac{1}{p} - 1} d\theta dr < \infty$$

in the sense that $f \in B^p$ if and only if F satisfy the above condition, where $\lim_{r\to 1} \text{Re } F(re^{i\theta}) = f(\theta)$ a.e. For an account of this space see [12], [13] and [14].

This characterization tells us that B^p in the boundary value of the analytic functions satisfying the above condition, a real characterization.

Theorem C will help us to have a better understanding of these Besov-Bergman spaces.

Finally, we mention that the same technique used to prove Theorem C can easily be used to show that

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|}{|x - y|} \, dydx < \infty.$$

As a consequence of the boundedness of this integral, any f in B^p satisfies Dini's condition and therefore the almost everywhere convergence is readily established, see [10]. Also other consequences of this technique will follow.

We would like to thank Professor Mitchell Taibleson with whom we had several conversations about these spaces.

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This paper was presented at the Seventh Symposium in Real Analysis, held at the St. Olaf College, Northfield, Minnesota, July 25-29, 1984.