α-variation and transformation into C^n functions

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The α-variation of a function $f\colon H \to \mathbb{R}$ $(H \subset \mathbb{R})$ is defined by

$$V_\alpha(f,H) = \sup \sum_{i=1}^{n} \left| f(b_i) - f(a_i) \right|^\alpha ,$$

where $\{[a_i,b_i]\}_{i=1}^n$ is an arbitrary system of non-overlapping intervals with $a_i, b_i \in H$ $i=1,2,\ldots,n$.

It is well-known that a continuous function f defined on $[a,b]$ can be transformed into a Lipschitz function by an inner homeomorphism if and only if $V_1(f,[a,b]) < \infty$. More generally, for every $\alpha \geqq 1$, f can be transformed into a Lipschitz $\frac{1}{\alpha}$ function if and only if $V_\alpha(f,[a,b]) < \infty$. More precise results concerning transformation of continuous functions of bounded 1-variation are due to A.M. Bruckner and C. Goffman [1]. They prove that f can be tranformed into a function with bounded derivative if and only if f is of bounded 1-variation, and it can be transformed into a C^1 function if and only if, in addition, the image of the set K_f of points of varying monotonicity is of measure zero. A point x is called a point of varying monotonicity of f if there is no neighbourhood of x on which f is strictly monotonic or constant.

In this paper we give an analogous characterization of those functions which can be transformed into a C^n function or into a function with bounded n^{th} derivative. We prove that if $n>1$ then f can be transformed into a C^n function if and only if f is continuous and $V_{1/n}(f,K_f) < \infty$. We get the same characterization for functions which can be transformed into a function with bounded n^{th} derivative, or into a function g with $g^{(n-1)} \in \mathrm{Lip}\ 1$. Hence, the case $n>1$ is different from that of $n=1$.

In order to formulate the precise results concerning the classes $\mathrm{Lip}\ s$ and C^s for every $s>0$, we need the following definitions.

For $\alpha>0$, $CBV_\alpha = CBV_\alpha[a,b]$ denotes the class of those functions $f \in C[a,b]$ for which $V_\alpha(f,K_f) < \infty$.

(It can be shown that

$$V_\alpha(f,K_f) = V_\alpha(f,[a,b]) \qquad \text{for every } \alpha \geqq 1,$$

hence for $\alpha \geqq 1$ CBV_α is the class of continuous functions with finite α-variation over $[a,b]$. If $0<\alpha<1$ then $V_\alpha(f,[a,b]) = \infty$ for every non-constant $f \in C[a,b]$.)

The strong α-variation of $f\colon H \to R$ is defined by

$$SV_\alpha(f,H) = \lim_{d \to 0+} V_\alpha^d(f,H) \quad ,$$

where $V_\alpha^d(f,H)$ is the supremum of those sums $\sum_{i=1}^{n} \left| f(b_i) - f(a_i) \right|^\alpha$ in which $[a_i,b_i]$ are non-overlapping intervals with $a_i, b_i \in H$ and $b_i - a_i \leqq d$.

The class SBV_α is defined as the family of those $f \in C[a,b]$ for which $SV_\alpha(f,K_f) < \infty$.

It is easy to show that $SBV_\alpha \subset CBV_\alpha$ for every $\alpha > 0$.

For $0 < s \leqq 1$ we denote by $\mathrm{Lip}\ s$ the class of those functions f defined on $[a,b]$ for which there is $K>0$ such that $\left| f(y) - f(x) \right| \leqq K|y-x|^s$ for every $x,y \in [a,b]$. If k is a positive integer and $k < s \leqq k+1$ then we denote by $\mathrm{Lip}\ s$ the class of all k times differentiable functions f defined on $[a,b]$ with $f^{(k)} \in \mathrm{Lip}(s-k)$.

For $0 \leqq s < 1$, C^s will denote the class of functions f defined on $[a,b]$ and satisfying the following condition.

For every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\left| f(y) - f(x) \right| \leqq \varepsilon \left| y-x \right|^s$$

whenever $x,y \in [a,b]$ and $|y-x| < \delta$.

Clearly, $C^0 = C[a,b]$. If k is a positive integer and $k \leqq s < k+1$ then we denote by C^s the class of all k times differentiable functions f defined on $[a,b]$ with $f^{(k)} \in C^{s-k}$.

<u>Theorem</u>. If $s > 1$ and $\alpha = 1/s$ then for every function f defined on $[a,b]$ the following are equivalent.

/i/ $f \in CBV_\alpha$.

/ii/ $f \in SBV_\alpha$.

/iii/ There is a homeomorphism φ of $[a,b]$ onto itself such that $f \circ \varphi \in \mathrm{Lip}\ s$.

/iv/ There is a homeomorphism φ of $[a,b]$ onto itself such that $f \circ \varphi \in C^s$.

If $s > 1$ is an integer, then these are also equivalent to the following.

/v/ There is a homeomorphism φ of $[a,b]$ onto itself such that $f \circ \varphi$ has bounded s^{th} derivative.

The results of Bruckner and Goffman show that the total equivalence of these statements for $s = 1$ does not hold. However, we have the following conditions for every $s > 0$.

<u>Theorem</u>. Let s and α be positive numbers with $s=1/\alpha$. A function f defined on $[a,b]$ belongs to CBV_α if and only if there is a homeomorphism φ of $[a,b]$ onto itself such that $f \circ \varphi \in \mathrm{Lip}\ s$.
$f \in SBV_\alpha$ if and only if there is a homeomorphism φ of $[a,b]$ onto itself such that $f \circ \varphi \in C^s$.

As for the class C^∞ of infinitely differentiable functions, we have the following

Theorem. For every f defined on $[a,b]$ the following are equivalent.

/i/ $f \in CBV_\alpha$ for every $\alpha > 0$.

/ii/ $f \in SBV_\alpha$ for every $\alpha > 0$.

/iii/ There is a homeomorphism φ of $[a,b]$ onto itself such that $f \circ \varphi \in C^\infty$.

In particular, if f can be transformed into a C^s function for every $s > 0$ then f can be transformed into a C^∞ function.

Reference

[1] A.M. Bruckner and C. Goffman, Differentiability through change of variables, Proc. Amer. Math. Soc. 61 /1976/, 235-241.

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