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Vladimír Šverák, MFF UK, Sokolovská 83, 18600 Prague 8, Czechoslovakia

Two connected topologies on the real line

In this note we give answers to the following problems:

(A) Let T be a topology on the real line R satisfying the following conditions:

(a) any interval is T-connected

(b) any continuous function is T-continuous

(c) if $x \in R$ and U is any T-neighbourhood of x, then $\mathrm{cl}U$ contains some neighbourhood of x

(d) if U is T-open and $x \in R$ then $U+x$ is also T-open.

Is it true that any T-continuous function is continuous?

(B) If T satisfies (a),(b),(c) and

(e) any T-continuous function is of the second class,

is it true that any T-continuous function is continuous?

The problem (A) was posed by Professor Swiatkowski at the conference held at Lodz in the autumn 1981 and the solution of the problem (B) gives an answer to the problem (3) from [3]. We show that the answer is negative in both cases.

Lemma 1: Let $f: R \to R$ be an arbitrary function and let T be the topology on R induced by continuous functions and the function f. Then

(1) T satisfies (a) iff the graph of f is connected

(2) T satisfies (b)

(3) if there exists an interval J such that $f(I)=J$ for any interval I, then T satisfies (c)

(4) if f is additive i.e. $f(x+y)=f(x)+f(y)$ for any $x,y \in R$, then T satisfies (d)

(5) T satisfies (e) iff f is of the second class.

<u>Proof</u>:

(1) It is easy to verify that the mapping $h: x \longrightarrow (x, f(x))$ is a homeomorphism of (R,T) onto the graph of f. Thus (R,T) is connected iff the graph of f is connected. Hence the proof will be finished by showing that if some closed interval $I=[a,b]$ is not connected, then (R,T) is not connected. Let U_1, U_2 be T-open sets such that

$$I \cap U_i \neq \emptyset \quad (i=1,2)$$

$$I \subset U_1 \cap U_2 \;, \quad a \in U_1$$

$$I \cap U_1 \cap U_2 = \emptyset .$$

If $b \in U_2$, put

$$V_1 = (U_1 \cap (-\infty, b)) \cup (-\infty, a)$$

$$V_2 = (U_2 \cap (a, \infty)) \cup (b, \infty)$$

and if $b \in U_1$, put

$$V_1 = U_1 \cup (R-I)$$

$V_2 = U_2 \cap (a,b)$

and we see that R is not T-connected.

The proof of the remaining parts of the lemma is easy. The following proposition shows that the answer to the problem (A) is negative.

Proposition 1: There exists a function $f:R\to R$ satisfying the following conditions:

(1) the graph of f is connected

(2) $f(I)=R$ for any interval I

(3) $f(x+y)=f(x)+f(y)$ for any $x,y\in R$.

Proof: See [1]

The following proposition shows that the answer to the problem (B) is negative.

Proposition 2: There exists a function $f:R\to R$ satisfying the following conditions:

(1) f is of the second class

(2) $f(I)=[0,1]$ for any interval I

(3) the graph of f is connected.

Proof: First we prove the following

Lemma 2: If $I\subset R$ is a closed interval and $D\subset I$ is a perfect nowhere dense set, then there exists a function $g:D\to[0,1]$ such that

(1) g is of the second class

(2) for any compact set $K\subset I\times[0,1]$ such that $p(K)=I$

there exists $t\in D$ such that $(t,g(t))\in K$.

Here $p:R^2\to R$ denotes the projection $(x_1,x_2)\to x_1$.

To prove the lemma denote by h a continuous mapping of D onto the compact metric space $X=\{K,\ K\subset I\times[0,1]\ ,K$ is compact and $p(K)=I\}$.(The metric ρ on X is defined by $\rho(K_1,K_2)=\max(\ \sup d(z,K_2)\ ,\ \sup d(z,K_1))$. See [4].)

For each positive integer n find open intervals $I_1^n,\ \ldots\ ,I_{p_n}^n$ and numbers $s_1^n,\ \ldots\ ,s_{p_n}^n$ such that

$$\bigcup_{j=1}^{p_n} I_j^n \supset D$$

$$s_j^n\in I_j^n\cap D \quad (j=1,\ \ldots\ ,p_n)$$

$$I_j^n\cap I_i^n\cap D \;=\emptyset \quad (j\neq i\ ,\ j,i=1,\ \ldots\ ,p_n)$$

$$|I_j^n|<2^{-n} \quad (j=1,\ \ldots\ ,p_n)$$

and for $t\in I_j^n\cap D$ put $g_n(t)=\inf\{x_2,(s_j^n,x_2)\in h(t)\}$.

We show that the function $g(t)=\limsup_{n\to\infty} g_n(t)$ satisfies the conditions (1),(2). Clearly g is of the second class since the functions g_n are lower-semicontinuous. Let K be a compact subset of $I\times[0,1]$ such that $p(K)=I$ and let $t\in D$ be such that $K=h(t)$. Find a sequence $\{k_n\}$ such that $t\in I_{k_n}^n$ for each n and denote $t_n=s_{k_n}^n$. Let $\{g_{n_i}\}$ be a subsequence of the sequence $\{g_n\}$ such that $g(t)=\lim_{i\to\infty} g_{n_i}(t)$. By definition of $g_n(t)$, $(t_n,g_n(t))\in K$ and therefore $(t,g(t))=\lim_{i\to\infty}(t_{n_i},g_{n_i}(t))\in K$.

Now we easily finish the proof of the proposition 2. Let I_n be a sequence of all closed intervals with rational endpoints. Find a sequence of perfect, nowhere dense sets such that $D_n \subset I_n$ and $D_i \cap D_j = \emptyset$ $(i \neq j\ ,\ i,j=1,2,\ \ldots)$. For each n use the lemma 2 with $D=D_n$, $I=I_n$ and denote by g_n the function g. Finally, put

$$f(x)= \begin{cases} g_n(x) & \text{if } x\in D_n \\ 0 & \text{if } x\in R-\bigcup_{n=1}^{\infty} D_n \end{cases}.$$

It is not difficult to verify that the function f satisfies the conditions (1),(2),(3).

In [2] it was proved that if T satisfies (c), then any T-continuous function having a dense set of points of continuity is continuous. It follows that we cannot construct a counterexample such that any T-continuous function is of the first class.

References:

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