RESEARCH ARTICLES Real Analysis Exchange Vol. 8 (1982-83)
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Algebra generated by derivatives

In this note we give an answer to the problem of characterizing the algebra generated by derivatives. In fact, we prove that each function of the first class can be expressed in the form $f'g'+h'$, which is the expression discussed in [2]. (We refer the reader to [2] for a more complete discussion of this problem and to [1] for a characterization of functions expressible in the form $fg'+h'$ $(f,g,h \in \Delta)$.) We prove slightly more, namely:

<u>Theorem</u>. Whenever $u: R \to R$ is a function of the first class, there are functions f, g and h possessing finite derivative everywhere such that $u = f'g'+h'$. Moreover, one can find this representation such that g' is bounded and h' is a Lebesgue function and, in case u is bounded, such that f' and h' are also bounded.

Recall that $x \in R$ is said to be a Lebesgue point of a function $\varphi: R \to R$ if $\lim\limits_{y \to x} (y-x)^{-1} \int_x^y |\varphi(t) - \varphi(x)|\,dt = 0$ and that φ is said to be a Lebesgue function if each $x \in R$ is a Lebesgue point of φ. Lebesgue functions are approximately continuous and they are derivatives of their indefinite integrals. Every bounded, approximately continuous function is a Lebesgue function. Hence for bounded functions these two notions coincide.

We will use the in-between theorem for Lebesgue functions (see Theorem 3.10 from [5]), but we will need only the following two special cases.

If B is a measurable set, if F_i are pairwise disjoint, closed sets contained in the set of those points of B that are density points of B and if $c_i \in R$ $(i = 1, \ldots, n)$, then there is a Lebesgue function φ such that $\varphi(x) = c_i$ for $x \in F_i$, $\varphi(x) = 0$ for $x \notin B$ and $|\varphi(x)| \leqq \max|c_i|$ for $x \in R$. (This is also an easy consequence of Theorem 7 from [6].)

If ψ is a function of the first class defined on R and if $|E| = 0$, then there is a Lebesgue function φ such that $\varphi(x) = \psi(x)$ for $x \in E$. (In the bounded case, this is one of the extension theorems of Petruska and Laczkovich [4].) To motivate our first lemma, we note that the statement of the Theorem can be read as follows: If u is a function of the first class, then there are differentiable functions f and g such that (g' is bounded and) $u - f'g'$ is a Lebesgue function. This formulation leads naturally to an attempt to approximate u by a product of derivatives in the L_1-norm and then to sum such approximations on a sequence of disjoint subsets.

<u>Lemma 1</u>. Assume that v is a measurable function, A is a nonempty, bounded, measurable set and $|v| \leq c < \infty$ on A. Then for every $\varepsilon > 0$ there are functions f and g possessing bounded, approximately continuous derivatives such that

$$\{x; f'(x) \neq 0\} \cup \{x; g'(x) \neq 0\} \subset A,$$

$$|f'| \leq \max(c, c^{1/2}), \quad |g'| \leq \min(1, c^{1/2}),$$

$$|f| \leq \varepsilon, \quad |g| \leq \varepsilon, \quad \text{and}$$

$$\int_A |v(t) - f'(t)g'(t)|dt \leq \varepsilon.$$

Proof. Write A as a union of nonempty, disjoint, measurable sets $A_1, \ldots, A_n$ such that $\operatorname{diam} A_i \leq \varepsilon/(3\max(1,c))$ and $\sup\{|v(x) - v(y)|; x, y \in A_i\} \leq \varepsilon/(3|A| + 1)$ for $i = 1, \ldots, n$. Let P_i and Q_i be disjoint, closed subsets of the set of those points of A_i that are points of density of A_i such that $|P_i| = |Q_i|$ and $|A_i - (P_i \cup Q_i)| \leq \varepsilon/(3n \max(1,c))$. Choose $x_i \in A_i$ and let $a_i = \max(|v(x_i)|, |v(x_i)|^{1/2})$ while $b_i = \min(1, |v(x_i)|^{1/2}) \operatorname{sgn} v(x_i)$. Let φ and ψ be Lebesgue functions such that $\varphi(x) = a_i$ and $\psi(x) = b_i$ for $x \in P_i$, $\varphi(x) = -a_i$ and $\psi(x) = -b_i$ for $x \in Q_i$, $|\varphi| \leq \max|a_i|$, $|\psi| \leq \max|b_i|$, and $\varphi(x) = \psi(x) = 0$ for $x \notin A$.

Let f and g be indefinite integrals of φ and ψ respectively such that $f(0) = g(0) = 0$. Obviously $|\int_{A_i} \varphi(t)dt| = |\int_{A_i - (P_i \cup Q_i)} \varphi(t)dt| \leq \max(c, c^{1/2}) \cdot \varepsilon/(3n\max(1,c)) \leq \varepsilon/(3n)$. Hence $|f(x)| \leq \varepsilon/3 + 2\max(c, c^{1/2})\max(\operatorname{diam} A_i) \leq \varepsilon$ and, similarly, $|g(x)| \leq \varepsilon$ for each $x \in R$. Clearly $\int_A |v(t) - f'(t)g'(t)|dt \leq 2nc\,\varepsilon/(3n\max(1,c)) + |A|\varepsilon/(3|A| + 1) \leq \varepsilon$ and the remaining statements of the lemma are obvious.

To sum up the different approximations, we need to interchange infinite sums and differentiation. For that we will use the following lemma which might also be of independent interest.

<u>Lemma 2</u>. Assume that (H_n) is a sequence of pairwise disjoint, compact subsets of the real line and that (K_n) is a sequence of nonnegative numbers such that $\sum K_n \chi_{H_n}$ (where χ_A is the characteristic function of A) is a function of the first class. Then there is a sequence (ϵ_n) of positive numbers such that the following two statements hold.

(1) Whenever $f_1, f_2, \ldots$ are differentiable functions, $|f_n'| \leq K_n$, $\{x; f_n'(x) \neq 0\} \subset H_n$ and $|f_n| \leqq \epsilon_n$ $(n = 1,2,\ldots)$, then $f = \sum f_n$ is well-defined, differentiable and $f' = \sum f_n'$.

(2) Whenever $w_1, w_2, \ldots$ are approximately continuous functions, $|w_n| \leq K_n$, $\{x; w_n(x) \neq 0\} \subset H_n$ and $\int_{H_n} |w_n| \leq \epsilon_n$ $(n = 1,2,\ldots)$, then $w = \sum w_n$ is a Lebesgue function.

<u>Proof</u>. Since $\sum K_n \chi_{H_n}$ is of the first class, there is a sequence (Q_n) of compact sets such that for each number r the set $\{x; \sum K_n \chi_{H_n} < r\}$ is a union of a subsequence of (Q_n). Put

$$\tilde{H}_n = \bigcup_{m<n} H_m \cup \bigcup_{m<n,\ Q_m \cap H_n = \emptyset} Q_m \quad \text{and} \quad \epsilon_n = 2^{-n} \min(1, \operatorname{dist}^2(H_n, \tilde{H}_n)).$$

To prove (1) we note that $f = \sum f_n$ is obviously well-defined, hence it suffices to show that $f'(x) = \sum f_n'(x)$ for each $x \in R$.

If $x \in H_k$ for some k, then for each $m > k$ and each $y \in R$ $|f_m(y) - f_m(x)| \leq 2^{-m+1}(y-x)^2$, since $(x,y) \cap H_m = \emptyset$ implies that the left hand side equals zero and since $(x,y) \cap H_m \neq \emptyset$ implies $\operatorname{dist}(H_m, \tilde{H}_m) \leq |y-x|$, hence $|f_m(y) - f_m(x)| \leq 2\,\epsilon_m \leq 2^{-m+1}(y-x)^2$. This inequality shows that $(\sum_{m=k+1}^{\infty} f_m)'(x) = 0 = \sum_{m=k+1}^{\infty} f_m'(x)$, and consequently $f'(x) = \sum f_n'(x)$.

If $x \in R - \bigcup H_n$ and if r is any positive number, we may find p such that $x \in Q_p$ and $\sum K_n \chi_{H_n} < r$ on Q_p. Then for each $m > p$ and each $y \in R$

$H_m \cap Q_p \neq \emptyset$ implies $K_m < r$, and hence

$$|f_m(y) - f_m(x)| \leq r|H_m \cap (x,y)|, \text{ while}$$

$H_m \cap Q_p = \emptyset$ and $(x,y) \cap H_m \neq \emptyset$ implies $\operatorname{dist}(H_m, \tilde{H}_m)$

$\leq |y-x|$, and hence $|f_m(y) - f_m(x)| \leq 2\epsilon_m \leq 2^{-m+1}(y-x)^2$, and finally $(x,y) \cap H_m = \emptyset$ implies $|f_m(y) - f_m(x)| = 0$.

Since $(x,y) \cap H_m = \emptyset$ for y sufficiently close to x and $m \leq p$,

$$\limsup_{y \to x} |y-x|^{-1} |\sum (f_n(y) - f_n(x))| \leq \limsup_{y \to x} (r + |y-x|) = r.$$

This shows that $f'(x) = 0 = \sum f_n'(x)$.

To prove (2), let $x \in R$ and let $w = u + v$, where $u = \sum_{n, x \in H_n} w_n$ and $v = \sum_{n, x \notin H_n} w_n$. Since u is bounded and approximately continuous, x is a Lebesgue point of u. Next we note that (1) implies that $|v|$ is a derivative, which together with $v(x) = 0$ proves that x is a Lebesgue point of v. Hence x is a Lebesgue point of w.

Finally, we need a decomposition lemma which will allow us to combine the above statements.

<u>Lemma 3</u>. Whenever u is a function of the first class, there are a function v of the first class, a sequence (H_n) of pairwise disjoint, compact sets and a sequence (c_n) of positive numbers such that

(i) $u - v$ is a Lebesgue function,

(ii) v is approximately continuous at every point of $\cup H_n$,

(iii) $v(x) = 0$ whenever $x \in H_n$ is not a point of density of H_n,

(iv) $|v| \leqq \Sigma\, c_n \chi_{H_n}$,

(v) $\Sigma\, c_n \chi_{H_n}$ is a function of the first class, and

(vi) v is bounded provided that u is bounded.

<u>Proof</u>. Let φ_1 be a Lebesgue function which agrees with u on a dense set containing all points at which u is not approximately continuous. (If u is bounded, we take φ_1 to be bounded also.) Let $v_1 = u - \varphi_1$. Since the function $\log|v_1|$ is of the first class on the space $X = \{x; v_1(x) \neq 0\}$, there is a function $g : X \to R$ of the first class such that $g(X)$ is an isolated subset of R and $|\log|v_1| - g| \leq 1$ on X (see [3], §31, VIII, Theorem 3). Using that X is an F_σ subset of R which does not contain an interval (and hence is a zero dimensional space), we see that X can be written as a union of a sequence (H_n) of

pairwise disjoint, compact sets such that g is constant on each H_n (cf. [3], §30,V). Let $w = \exp(g+1)$ on X, $w = 0$ on $R - X$ and let c_n be positive numbers such that $w = \sum c_n \chi_{H_n}$. Then (v) holds since $w \geq 0$ and for each $a > 0$

$\{x \in R; w(x) > a\} = \{x \in X; g(x) > -1 + \log a\}$ while

$\{x \in R; w(x) < a\} = \{x \in R; |v_1(x)| < ae^{-2}\} \cup \{x \in X; g(x) < -1 + \log a\}$.

Let E be a set of measure zero containing all points of each H_n that are not density points of H_n and containing all points at which v_1 is not approximately continuous. Let φ be a Lebesgue function agreeing with v_1 on E and let

$\varphi_2 = \max\{\min[\varphi, \max(v_1, 0)]; \min(v_1, 0)\}$. Then $|\varphi_2| \leq |\varphi|$. Hence every $x \in R$ with $\varphi(x) = 0$ is a Lebesgue point of φ_2. If $\varphi(x) \neq 0$, then v_1 is approximately continuous at x, and hence φ_2 is approximately continuous at x, which together with $|\varphi_2| \leq |\varphi|$ shows that x is a Lebesgue point of φ_2. Since $\min(v_1, 0) \leq \varphi_2 \leq \max(v_1, 0)$ and since $|v_1| \leq w = \sum c_n \chi_{H_n}$, the function $v = v_1 - \varphi_2$ fulfils (iv). The other statements of the lemma are obvious.

<u>Proof of the theorem.</u> Let v, H_n and c_n be as in Lemma 3. For the sequences (H_n) and $K_n = 2\max(c_n, c_n^{1/2})$ we find positive numbers ϵ_n according to Lemma 2. For each n we use Lemma 1 with $A = H_n$ and $\varepsilon = \epsilon_n$ to construct functions f_n and g_n with the properties described there. From Lemma 2 we see that the functions $f = \sum f_n$ and $g = \sum g_n$ are well-defined, that $f' = \sum f_n'$ and $g' = \sum g_n'$ and that $v - f'g' = \sum (v - f_n'g_n')\chi_{H_n}$ is a Lebesgue function. (Here we use (iii) from Lemma 3 and

approximate continuity of f_n' and g_n' to deduce approximate continuity of $w_n = (v - f_n'g_n')\chi_{H_n}$.) Therefore, the desired representation is $u = f'g' + h'$, where $h' = (u - v) + (v - f'g')$.

I would like to express my thanks to the referee for finding and correcting many errors in the first version of the proof, to prof. A. Bruckner for nicer formulation of the assumption of Lemma 2 as well as for many other remarks and to the organizers and participants of the Esztergom meeting, since it was mainly my talk there and the discussions after it which led to a new (and, I believe, reasonably simple) version of the proof.

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Received October 27, 1982