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Differentiation and Lusin's Condition (N)

This paper deals with a problem mentioned by Professor D.W. Solomon; namely, whether two continuous functions can each satisfy Lusin's condition (N), be differentiable a.e. with identical derivatives a.e. and not differ from each other by a constant. That this can occur is shown in the example below. The functions in the example differ by a monotone function and Theorem 1 shows that a function which has a pair of this type also has a pair which differs from it by a monotone function. Theorem 2 shows that no function with a pair can be ACG .

Example: *There exist two continuous functions f_1 and f_2 which satisfy Lusin's condition (N), are differentiable a.e. with equal derivatives a.e., such that $f_1 - f_2$ is not identically constant.*

Proof: Note that each real number $x \in [0,1]$ can be written as $\Sigma x_i \cdot 16^{-i}$ where $0 \leq x_i < 16$ or, alternatively, as $\Sigma(\frac{1}{2}x_i) \cdot 8^{-i}$ where $0 \leq x_i < 16$ and each x_i is even. Let P be the set of all $x = \Sigma x_i \cdot 16^{-i}$ where $0 \leq x_i < 16$ and each x_i is even. Then P is perfect, of measure 0, and contained in [0,15/16]. If $x \in P$ and $x = \Sigma x_i \cdot 16^{-i}$, define $f_1(x)$ by $f_1(x) = \Sigma a_i \cdot 8^{-i}$

where $a_i = 6$ if 4 divides x_i, $a_i = 0$ otherwise; define $h(x) = 2\,\Sigma(\frac{1}{2}x_i) \cdot 8^{-i}$. Define f_1 and h on $[0, 15/16]$ by extending them linearly from P to the intervals contiguous to P. Since both f_1 and h are continuous on P they are continuous on $[0, 15/16]$. Note that $h(x)$ takes P onto $[0,2]$, is monotone non-decreasing and is constant on intervals contiguous to P. Define $f_2(x) = f_1(x) + h(x)$. Then both f_1 and f_2 are differentiable almost everywhere with $f_1' = f_2'$ a.e.. Now, $f_1(P)$ is clearly of measure 0 and since f_1 is linear on intervals contiguous to P, f_1 satisfies condition (N). If $y \in f_2(P)$, then

$$y = \Sigma(a_i + x_i)8^{-i} \quad \text{where } a_i = 6 \text{ if } x_i = 0,4,8 \text{ or } 12$$
$$\text{and } a_i = 0 \text{ if } x_i = 2,6,10, \text{ or } 14$$

Thus, the possible values of $a_i + x_i$ are 2, 6, 10, 14, or 18. Hence, $f_2(P)$ can be covered with 5^k intervals each of length at most $2 \cdot 8^{-k} + 6\sum_{k}^{\infty} 8^{-i} = 8^{-k} \cdot 62/7$.

It follows that $|f_2(P)| = 0$. Since f_2 is also linear on intervals contiguous to P, f_2 also satisfies condition (N).

Theorem 1. *If f_1 and f_2 are continuous functions, which satisfy conditions (N) and are differentiable almost everywhere with $f_1' = f_2'$ a.e., then there exists a continuous function f_3 which also satisfies condition (N) such that $f_3' = f_1'$ a.e. and $f_3 - f_1$ is monotone.*

Proof. Let f_1 and f_2 satisfy the hypotheses of the theorem and let $h = f_2 - f_1$. Assuming h is not already monotone, let $y_1 = \sup h$, $y_0 = \inf h$ and find x_0 and x_1 such that $f(x_0) = y_0$, $f(x_1) = y_1$. Without loss of generality, $x_0 < x_1$. For each $y \in [y_0, y_1]$, let $x(y) = \inf \{x \mid x \in [x_0, x_1] \text{ and } h(x) = y\}$. Define $g(x) = y$ if $x = x(y)$ and extend g continuously to the closure E of the set of $x(y)$ and then linearly to $[0,1]$ with $f(x) = y_0$ if $x < x_0$ and $f(x) = y_1$ if $x > x_1$. Since $h'(x) = 0$ a.e. and g agrees with h on E and is constant on each interval contiguous to E, it follows that $g'(x) = 0$ a.e. It is clear that $g(x)$ is monotone. Let $f_3(x) = f_1(x) + g(x)$. Then $f_3'(x) = f_1'(x)$ a.e. . Since $f_3(x) = f_2(x)$ at each point $x \in E$ and on each interval I_n contiguous to E, $f_3(x) = f_1(x) + C_n$, where C_n are appropriate constants; it follows that $f_3(x)$ satisfies condition (N).

Theorem 2. *If f_1 is ACG, f_2 is continuous and satisfies condition (N) and both f_1 and f_2 are differentiable a.e. with $f_1' = f_2'$ a.e., then $f_2 - f_1$ is identically constant.*

Proof. Suppose not and let $h = f_2 - f_1$. Construct g and f_3 as in Theorem 1. Then $f_1 + g = f_3$ and f_3 is both VBG and satisfies condition (N). By [1, Thm. 6.7, p.227], f_3 is ACG. Hence, g is ACG and since g is monotone, g is absolutely continuous. Since $g' = 0$ a.e., g is identically constant. But this is impossible unless h were constant and in that case $f_2 - f_1$ is constant.

Note: Theorems 1 and 2 can be proven in the same fashion using the approximate derivative rather than the ordinary derivative.

REFERENCES

1. S. Saks, Theory of the Integral, New York 1937.

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