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Richard J. Fleissner, Department of Mathematics,
Western Illinois University, Macomb, Illinois 61455
James Foran, Department of Mathematics, University
of Missouri-Kansas City, Kansas City, Missouri 64110

Transformations of Differentiable Functions

Let T denote the set of real valued functions defined on $[0,1]$ of the form $U \circ F$ where $F(x)$ is a differentiable function and U is a homeomorphism. Equivalently, T consists of the continuous functions which can be homeomorphically transformed into differentiable functions.

Differentiable functions are ACG* and thus satisfy Banach's condition (T_1) and Lusin's condition (N) . A reader unfamiliar with these conditions is referred to [2, Ch.9].

The main result of [1] is that the condition (S′) along with continuity characterize the class T .

A function is said to satisfy condition (S′) if for each open interval J contained in the range of F , there exists $\varepsilon_J > 0$ such that $J \subset F(E)$ implies $|E| > \varepsilon_J$

A function is said to satisfy condition (S″) if for each open interval J contained in the range of

F , $J \subset F(E)$ implies $|E| > 0$.

Differentiable functions must satisfy (S″) since they satisfy condition (N) . Moreover, (S″) is preserved by outer homeomorphisms, as is (S′) . Thus Lebesgue's singular function provides an example of a continuous function which does not belong to T since it maps the cantor set onto [0,1] . Continuity is clearly necessary for a function to belong to T . To show the necessity of (S′) , we assume that F does not satisfy this condition and note that if G is any homeomorphism, then $G \circ F$ does not satisfy it either. It is then shown that if $G \circ F$ does not satisfy (S′) , then it cannot satisfy both (T_1) and (N) and thus cannot be differentiable. To show sufficiency, we define

$$\varepsilon(J) = \inf\{|E| : J \subset F(E)\}$$

and set $G(y) = \varepsilon([0,y])$. Then G is a homeomorphism and it is the condition (S′) which guarantees that G is strictly increasing. Then $G \circ F$ satisfies a Lipschitz condition of order one and thus is differentiable almost everywhere and the image Q of the set of points of nondifferentiability of $G \circ F$ is of measure zero. Using a result of Zahorski [3], we define H to be a differentiable homeomorphism whose derivative is zero on the set Q . Then, $H \circ G \circ F$ is shown to be everywhere differentiable.

We construct an example of an ACG function which does not belong to $\mathcal{T}$. This function satisfies (N) and thus (S″) and this shows that (S′) and (S″) are not equivalent.

We further note that ACG* functions belong to $\mathcal{T}$ since ACG* ⇒ (T_1) and (N) ⇒ (S′) ⇔ $\mathcal{T}$.

A function is then constructed which belongs to $\mathcal{T}$ but does not satisfy Banach's condition (T_2) . Functions in $\mathcal{T}$ are shown to be differentiable on a set of positive measure in every interval. A final example is given of a function in $\mathcal{T}$ which fails to be differentiable on a set of positive measure.

References

1. R.J. Fleissner and J. Foran, Transformations of Differentiable Functions (submitted to Colloquium Mathematicum).

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3. Z. Zahorski, Sur la première dérivée, Trans. Amer. Math. Soc. 69 (1950), p. 1-54.

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