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A Local Characterization of Darboux(B) Functions. The Semicontinuity of Monotone Functions.

In 1987 Prof. B. Ricceri from Catania (Italy), in a private letter, raised the following problem:

> Let X be a connected, locally connected and complete metric space, and let f be a real-valued function on X such that, for every $r \in \mathbb{R}$, the set $f^{-1}(r)$ is non-empty and arcwise connected. Find some sufficient (and possibly necessary) conditions under which the function f is lower semicontinuous.

Our paper includes a full answer to the question raised by B. Ricceri. At the same time, we discuss important problems of Darboux points, investigated lately very intensely by many mathematicians. Our results are connected with the considerations included in [BB].

The notion of a Darboux point (for real functions of a real variable) was introduced for the first time in paper [BC]. In paper [BB] the authors introduced the notion of real Darboux(B) functions and gave a local characterization of these transformations. Some generalizations of these results are included in [RG]. The consideration of problems of a local characterization of transformations connected with the notion of connectedness can be found, among others, in [JL], [JJ], [GNK], [LS], [RP].

In papers [RG] and [BB] this notion referred to Darboux(B) functions. In our paper the term Darboux(B) will be understood in a bit more general sense than in [RG] and [BB]: Let $f : X \to \mathbb{R}$ where X is some metric space, and let B be a family of connected sets, covering X, (i.e., $X = \bigcup_{A \in B} A$). We say that f is a Darboux(B) function if $f(\bar{U})$ is a connected set for $U \in B$ [1]

Similarly as in [BB] and [RG], as concerns the family B, we shall consider two conditions:

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[1] The difference between our definition and that contained in [RG] and [BB] lies in the fact that, in these papers, B has assumed to be a base of X. In our considerations this assumption is dispensable.

- A family B (in a euclidean space X) is said to satisfy condition $(*)$ provided any translation of an element of B is in B.

- A family B (in a metric space X) satisfies condition $(**)$ provided that, for any $x \in X$ and $U \in B$, $x \in \bar{U}$, there exists $V \in B$ such that $x \in \bar{V}$ and $\bar{V} \setminus \{x\} \subset U$.

If B contains singletons, then conditions $(**)$ can say nothing in the sense that $\bar{V} \setminus \{x\} = \phi$. So, if we assume that a family B satisfies condition $(**)$, then we understand that $\{x\} \notin B$, for any $x \in X$.

Let X be any metric space, and let B be a family of connected sets; moreover, let $f : X \to \mathbb{R}$. We shall say that $x_0 \in X$ is a lower - Darboux(B) point (an ℓ-Darboux(B) point) of f if, for any $U \in B$ such that $x_0 \in \bar{U}$, any sequence $\{x_n\} \subset U$ such that $x_0 = \lim_{n \to \infty} x_n$ and any real numbers α, β, if $f(x_n) \leq \beta < \alpha < f(x_0)$ for $n = 1, 2, \ldots$, then $\alpha \in f(\bar{U})$ ($\alpha \in f(U)$). In a similar way we define an upper-Darboux(B) point (a u-Darboux(B) point) of f.

We say that $x_0 \in X$ is a Darboux(B) point (an s-Darboux(B) point) of f if it is simultaneously a lower- and upper-Darboux(B) point (an ℓ- and u-Darboux(B) point) of f.

We say that f is an upper-Darboux(B) function (a lower-Darboux(B) function, a u-Darboux(B) function, an ℓ-Darboux(B) function) if each point $x \in X$ is an upper-Darboux(B) point (a lower-Darboux(B) point, a u-Darboux(B) point, an ℓ-Darboux(B) point) for f.

The above definitions are similar to the considerations contained in [BB]. The form of these definitions is analogous to the definitions of a Darboux point in [JL] and Darboux points of the first, second and third kinds from [RP].

Proposition 1 *Let $f : \mathbb{R} \to \mathbb{R}$. Then the following conditions are equivalent:*

(i) x_0 is a Darboux point (in the sense of the definition adopted in [BC]);

(ii) x_0 is an s-Darboux(B) point where $B = \{(p,q) : p, q \in \mathbb{R}\}$;

(iii) x_0 is a Darboux(B) point where $B = \{(p,q) : p, q \in \mathbb{R}\}$.

The next Proposition is analogous to Theorem 1 of [BB].

Proposition 2 *Let X be a euclidean space and B a base of connected sets for X which satisfies conditions $(*)$ and $(**)$. Then a function $f : X \to \mathbb{R}$ is Darboux(B) if and only if each point $x_0 \in X$ is a Darboux(B) point for f.*

An illustration of the essentiality of the assumptions of Proposition 2 is the following example.

Example. Let $X = \mathbb{R}$ and $B = \{(p,q) : p, q \in \mathbb{R} \wedge q \neq 0\}$. Define $f : \mathbb{R} \to \mathbb{R}$ by letting

$$f(x) = \begin{cases} \sin(1/x) + x + 1 & \text{for } x > 0, \\ 2 & \text{for } x = 0, \\ x & \text{for } x < 0. \end{cases}$$

Then f is lower- and upper-Darboux(B) at each $x \in X$, but f is not Darboux(B).

In many questions, it is essential to consider the Darboux properties for functions of the first class of Baire (cf. e.g. [ZZ], [BC], [BB]). Hence it is interesting to attempt a "local characterization" of Darboux functions of Baire class I (cf. e.g. [JY], [RP]). The theorem below, because of its character and the possibility of choosing the family B in different ways, has in its scope a rather considerable group of different generalizations of Darboux functions.

Lemma 1 *Let $f : X \to \mathbb{R}$ be a lower-Darboux(B) function, where B satisfies condition $(**)$. Then, for each $U \in B$:*

$$\textit{if } f(U) \subset (-\infty, \beta], \textit{ then } f(\bar{U}) \subset (-\infty, \beta].$$

Of course, an analogous implication holds for the case of upper-Darboux(B) functions, too.

Proof is obvious.

Theorem 1 *Let X be a complete space and let B be a family of arcwise connected sets, covering X, satisfying condition $(**)$ and such that each arc $\mathcal{L} \subset X$ belongs to B. Moreover, let $f : X \to \mathbb{R}$ be a function of the first class of Baire. Then f is a Darboux(B) function if and only if each point $x \in X$ is a Darboux(B) point of f.*

Proof. Necessity. Let $x_0 \in X$ and let U be an element of the family B, such that $x_0 \in \bar{U}$. Moreover, let $\{x_n\}$ be a sequence of elements from U, such that $\lim_{n\to\infty} x_n = x_0$, and let α, β be real numbers for which $f(x_n) \leq \beta < \alpha < f(x_0)$ $(n = 1, 2, \ldots)$. This means that, for some n_0, $f(x_{n_0}), f(x_0) \in f(\bar{U})$. In view of the connectedness of $f(\bar{U})$, it may be inferred that $\alpha \in f(\bar{U})$. It is proved similarly that x_0 is an upper-Darboux(B) point of f.

Sufficiency. Suppose that f is not a Darboux(B) function. Then there exists a set $C \in B$ such that $f(\bar{C})$ is not a connected set. Thus

$$f(C) \text{ is not a connected set.} \tag{1}$$

Indeed, as $f(\bar{C})$ is not a connected set, there is $\gamma \in \mathbb{R} \setminus f(\bar{C})$ such that

$$f(\bar{C}) \cap (-\infty, \gamma) \neq \phi \neq f(\bar{C}) \cap (\gamma, +\infty).$$

By Lemma 1 it is clear that $f(C) \cap (-\infty, \gamma) \neq \phi \neq f(C) \cap (\gamma, +\infty)$, which ends the proof of (1).

It is not hard to show that, under our assumptions,

$$f(y) \in \overline{f(\mathcal{L} \backslash \{y\})} \text{ for any } y \in X \text{ and any arc } \mathcal{L} \imath X \text{ with end at } y. \tag{2}$$

In view of (1), we may deduce that there exists an arc $\mathcal{L}$ with ends at points p and q, such that $f(\mathcal{L}) = E \cap F$ where E and F are non-empty separated sets. Denote $E' = f^{-1}(E) \cap \mathcal{L}$ and $F' = f^{-1}(F) \cap \mathcal{L}$ and let D be the boundary of E' in $\mathcal{L}$ as a subspace. Then D is a non-empty closed set, thus $f_{|_D}$ possesses some point of continuity z. Assume, with no loss of generality, that $z \in E'$. Then there exists $\xi > 0$ such that $(f(z) - \xi, f(z) + \xi) \cap \bar{F} = \phi$. Consequently, let U be a neighborhood of z such that $f(U \cap D) \imath (f(z) - \xi, f(z) + \xi)$. This means that

$$U \cap D \cap F' = \phi. \tag{3}$$

Let us introduce the notation: for an arbitrary arc $\mathcal{K}$, the symbol $L_{\mathcal{K}}(a, b)$ denotes an arc with ends a and b, contained in $\mathcal{K}$.

Coming back to our proof, let m and n be elements of the arc $\mathcal{L}$, such that $z \in L_{\mathcal{L}}(m, n) \imath U$ (here we demand that $m \neq z \neq n$ whenever $p \neq z \neq q$). Let further $z_1 \in L_{\mathcal{L}}(m, n) \imath F'$ and adopt $\mathcal{L}^* = L_{\mathcal{L}}(z, z_1)$. Let h be a homeomorphism, mapping $[0, 1]$ onto $\mathcal{L}^*$, such that $h(0) = z$ and $h(1) = z_1$. Let $E'' = h^{-1}(E' \imath \mathcal{L}^*)$ and $F'' = h^{-1}(F' \cap \mathcal{L}^*)$. In view of (3), z_1 belongs to the interior of F' and $\mathcal{L}$, therefore $1 \in \mathrm{Int} F''$. Let $\eta = \inf\{\beta < 1 : (\beta, 1] \imath F''\}$ and let $\eta' = h(\eta)$. It can be demonstrated that $\eta' \in U \cap D$. In virtue of (2), we may infer that $\eta' \notin E'$, thus $\eta' \in U \cap D \cap F'$, which contradicts (3). The contradiction obtained completes the proof of the theorem.

In many papers (cf. e.g. [KM], [CM], [GW]), the authors investigated different variants of generalizations of the notion of monotonicity to the case of transformations in topological spaces. The definition presented below is one of the acceptable versions of this notion.

Definition. Let $f : X \to Y$, where X is a metric space, and let B be a family of connected sets, covering X. We say that f is B-monotone if $f^{-1}(\beta) \in B$ for any $\beta \in Y$.

Of course, if B_0 is the family of all connected subsets of X, then B_0-monotonicity means weak monotonicity in the sense adopted in paper [KG], and if B^0 is the family of all continua, B^0-monotonicity is identical with Morrey monotonicity.

We shall now proceed to giving the basic answers to the problem posed by Prof. B. Ricceri:

Theorem 2 *Let X be a metric space, B_1 a family of connected sets, satisfying condition $(**)$, and B a family of connected sets containing B_1 and some base B_2*

of the space X. Moreover, let $f : X \to \mathbb{R}$ be a B_1 monotone function. Then f is lower- (upper-) semicontinuous if and only if f is a lower- (upper-) Darboux(B) function.

Proof. The proof will be carried out in the case of the "lower-" property only since, in the other case, it runs analogously.

Necessity. Let $x_0 \in X$. Then if $x_n \to x_0$, then, in view of the lower-semicontinuous of f, there exists no number $\beta < f(x_0)$ such that $f(x_n) \leq \beta$ for $n = 1, 2, \ldots$.

Sufficiency. Let β be an arbitrary real number. We shall demonstrate that $f^{-1}((-\infty, \beta])$ is closed. Suppose that it is not the case; hence there exists a point x_0 such that

$$x_0 \in \overline{f^{-1}((-\infty, \beta])} \backslash f^{-1}((-\infty, \beta]).$$

Thus, there is a sequence $\{x_n\}_{n=1}^{\infty} 1 f^{-1}((-\infty, \beta])$ such that x (with index 0) $= \lim_{n \to \infty} x_n$. Then $f(x_n) \leq \beta$ $(n = 1, 2, \ldots)$ and $f(x_0) > \beta$. Let α be a real number such that $\beta < \alpha < f(x_0)$. By Lemma 1, x_0 is not in the union $\overline{f^{-1}(\alpha)} \cup \overline{f^{-1}(\beta)}$; therefore there exists a neighborhood U of x_0 such that $x_0 \in U 1 \bar{U} 1 X \backslash (\overline{f^{-1}(\alpha)} \cup \overline{f^{-1}(\beta)})$ and U is in B. Then α is not in $f(\bar{U})$ and, moreover, there exists n_0 such that $x_n \in U$, for $n \geq n_0$; which is a contradiction with f being lower-Darboux(B) at x_0.

Making use of the well-known Mazuerkiewicz-Moore theorem and applying arguments analogous to those in the above theorem, one can show that:

Theorem 3 *Let X be a locally connected and complete metric space and let f be a real-valued function on X such that, for every $r \in \mathbb{R}$, the set $f^{-1}(r)$ is non-empty and arcwise connected. Then f is lower- (upper-) semicontinuous if and only if f is an ℓ-Darboux(B) (a u-Darboux(B)) function, where B stands for the family of all arcwise connected sets in X.*

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