Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 82–86

Hüseyin Bor, Department of Mathematics, Erciyes University, Kayseri 38039, Turkey, Mailing Address: P.K. 213, Kayseri 38002, Turkey

A Note on Absolute Nörlund Summability Factors

Let $\sum a_n$ be an infinite series with sequence of partial sums (s_n). By δ_n and t_n we denote the nth $(C, 1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} |\delta_n - \delta_{n-1}|^k < \infty. \tag{1}$$

Since $t_n = n(\delta_n - \delta_{n-1})$ (see [4]), condition (1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{2}$$

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0, \quad (n \geq 0). \tag{3}$$

The sequence-to-sequence transformation

$$z_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{4}$$

defines the sequence (z_n) of the Nörlund means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|N, p_n|$ if (see [5])

$$\sum_{n=1}^{\infty} |z_n - z_{n-1}| < \infty, \tag{5}$$

Key Words: Absolute summability, strong summability, summability factors, Nörlund methods.

Mathematical Reviews subject classification: 40D15, 40F05, 40G05

Received by the editors September 4, 1991

and it is said to be summable $|N, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{k-1} |z_n - z_{n-1}|^k < \infty. \tag{6}$$

In the special case when $p_n = 1$ and $P_n = n + 1$, the Nörlund mean reduces to the $(C, 1)$ mean and $|N, p_n|_k$ summability becomes $|C, 1|_k$ summability.

Varma [6] proved the following theorem concerning the $|C, 1|_k$ and $|N, p_n|_k$ summability methods.

Theorem 1 *Let $p_0 > 0$, $p_n \geq 0$ and let (p_n) be a nonincreasing sequence. Let $k \geq 1$. If $\sum a_n$ is summable $|C, 1|_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $|N, p_n|_k$.*

Quite recently the author proved the following theorem (see [1]).

Theorem 2 *Let (p_n) be a sequence as in Theorem 1. If*

$$\sum_{v=1}^{n} \frac{1}{v} |t_v| = O(X_n) \quad as \quad n \to \infty, \tag{7}$$

where (X_n) is a positive nondecreasing sequence and (l_n) is a sequence such that[1]

$$\sum_{n=1}^{\infty} n X_n |\Delta^2 l_n| < \infty \tag{8}$$

$$l_n X_n = O(1) \quad as \quad n \to \infty, \tag{9}$$

then the series $\sum a_n l_n P_n (n+1)^{-1}$ is summable $|N, p_n|$.

The aim of this paper is to generalize Theorem 2 for $|N, p_n|_k$ summability with $k \geq 1$. Now, we shall prove the following theorem.

Theorem 3 *Let (p_n) be a sequence as in Theorem 1 and let $k \geq 1$. If*

$$\sum_{v=1}^{n} \frac{1}{v} |t_v|^k = O(X_n) \quad as \quad n \to \infty, \tag{10}$$

and the sequences (X_n) and (l_n) are such that conditions (8), (9) of Theorem 2 are satisfied, then the series $\sum a_n l_n P_n (n+1)^{-1}$ is summable $|N, p_n|_k$.

[1] $\Delta^2 l_n = \Delta(\Delta l_n)$ *and* $\Delta l_n = l_n - l_{n+1}$.

It should be noted that if we take $k = 1$ in this theorem, then we get Theorem 2.

We need the following lemma for the proof of our theorem.

Lemma 4 ([1]) *Under the conditions of the theorem we have*

$$nX_n \Delta l_n = O(1) \quad as \quad n \to \infty \tag{11}$$

$$\sum_{n=1}^{\infty} X_n |\Delta l_n| < \infty. \tag{12}$$

Proof of the Theorem. By virtue of Theorem 1, we need only deal with special case in which $p_n \equiv 1$, that is we shall prove that $\sum a_n l_n$ is summable $|C,1|_k$, $k \geq 1$. Let T_n be the n-th $(C,1)$ mean of the sequence $(na_n l_n)$, that is

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v l_v. \tag{13}$$

Now, applying Abel's transformation, we have something similar to

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n-1} \Delta l_v (v+1) t_v + l_n t_n = T_{n,1} + T_{n,2}.$$

To complete the proof of the theorem it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |T_{n,i}|^k < \infty, \quad \text{for } i = 1, 2. \tag{14}$$

First note that

$$\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}|^k = O(1) \sum_{n=2}^{m+1} n^{-k-1} \left\{ \sum_{v=1}^{n-1} v |\Delta l_v| |t_v| \right\}^k.$$

When $k > 1$ with $\frac{1}{k} + \frac{1}{k'} = 1$, we apply Hölder's inequality to the right hand side. It turns into

$$O(1) \sum_{n=2}^{m+1} n^{-k-1} \sum_{v=1}^{n-1} (v |\Delta l_v| |t_v|)^k \times \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k/k'},$$

which, for any $k > 1$, is

$$O(1) \sum_{n=2}^{m+1} n^{-k-1} \left\{ \sum_{v=1}^{n-1} v |\Delta l_v| |t_v|^k \right\} \times O(n^{k-1})$$

through (11). Thus

$$O(1)\sum_{v=1}^{m} v|\Delta l_v||t_v|^k \sum_{n=v+1}^{m+1} = O(1)\sum_{v=1}^{m} v|\Delta l_v|\frac{1}{v}|t_v|^k$$

$$= O(1)\left\{\sum_{v=1}^{m-1} |\Delta(v|\Delta l_v|)|\sum_{r=1}^{v}\frac{1}{r}|t_r|^k + m(\Delta l_m)\sum_{v=1}^{m}\frac{1}{v}|t_v|^k\right\}$$

$$= O(1)\sum_{v=1}^{m-1} |\Delta(v|\Delta l_v|)|X_v + O(1)mX_m\Delta l_m$$

$$= O(1)\sum_{v=1}^{m-1} vX_v|\Delta^2 l_v| + O(1)\sum_{v=1}^{m-1}|\Delta l_{v+1}|X_v + O(1)mX_m\Delta l_m = O(1)$$

as $m \to \infty$, by virtue of (8), (10), (11), and (12). Since $l_n = O(1/X_n) = O(1)$, by (9), we have

$$\sum_{n=1}^{m}\frac{1}{n}|T_{n,2}|^k = \sum_{n=1}^{m}\frac{1}{n}|l_n t_n|^k = \sum_{n=1}^{m}|l_n|^{k-1}|l_n|\frac{1}{n}|t_n|^k$$

$$= O(1)\sum_{n=1}^{m}|l_n|\frac{1}{n}|t_n|^k = O(1)\sum_{n=1}^{m-1}|\Delta l_n|\sum_{v=1}^{n}\frac{1}{v}|t_v|^k + O(1)l_m\sum_{n=1}^{m}\frac{1}{n}|t_n|^k$$

$$= O(1)\sum_{n=1}^{m-1}|\Delta l_n|X_n + O(1)l_mX_m = O(1) \quad \text{as } m \to \infty,$$

by virtue of (9), (10), and (12). Therefore, we get that

$$\sum_{n=1}^{m}\frac{1}{n}|T_{n,i}|^k = O(1) \quad \text{as } m \to \infty, \quad \text{for } i = 1, 2.$$

This completes the proof of the theorem.

The author offers his sincerest gratitude to the referee for his kind interest and valuable suggestions for the improvement of this paper.

References

[1] H. Bor, *Absolute Nörlund summability factors*, Utilitas Math., (in press).

[2] D. Borwein and F.P. Cass, *Strong Nörlund summability*, Math Z., 103 (1968), 94-111.

[3] T.M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc., 7 (1957), 113-141.

[4] E. Kogbetliantz, *Sur les séries absolument sommables par la méthode des moyennes arithmétiques*, Bull. Sci. Math. (2), 49 (1925), 234-256.

[5] F.M. Mears, *Some multiplication theorems for the Nörlund mean*, Bull. Amer. Math. Soc., 41 (1935), 875-880.

[6] S.M. Varma, *On the absolute Nörlund summability factors*, Riv. Mat. Univ. Parma (4), 3 (1977), 27-33.