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Positive Linear Functionals on Spaces of Continuous Functions

1. Introduction

In [9] Hausdorff defines a complete ordinary function system Ω on a space X as a linear lattice of continuous functions containing the constants which is uniformly closed, which is a ring, and which is closed under inversion, i.e., if $f \in \Omega$ and $f > 0$, then $1/f \in \Omega$ (here $f > 0$ means that $f(x) > 0$ for all $x \in X$ and $f \geq 0$ means that $f(x) \geq 0$ for all $x \in X$). In particular, each space $C(X)$ of all continuous functions on a topological space is a complete ordinary function system (abbreviated cofs). These systems of functions have been studied by many other authors and we shall refer to some of them in this paper.

If Ω is a cofs, then the bounded functions in Ω form a real Banach algebra under the uniform norm that we shall denote by Ω^*. A representation by measures of the dual space of this Banach space has been obtained by Alexandroff in [1].

The aim of this paper is to represent all positive functionals defined on a cofs Ω by means of integrals. This representation was given by Hewitt in [12], Theorems 13 and 18, when Ω is $C(X)$ for X a realcompact space. Cater in [3] gives a representation of all positive linear functionals defined on $B(X)$, the set of all Baire functions on a realcompact space X, as finite sums of Riesz Homomorphisms. Finally, Tucker in [18] considers a cofs Ω and obtains a representation of all positive linear functionals defined on $B_1(\Omega)$, the set of all pointwise limits of sequences in Ω, as sums of a finite number of Riesz homomorphisms.

2. Preliminaries

$\mathbb{N}$ (resp. $\mathbb{R}, \mathbb{Q}$) will denote the set of all natural numbers (resp. real numbers, rational numbers).

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By space we mean a completely regular Hausdorff space. The closure of a subset A of X will be denoted by cl A or, when there is possibility of confusion, by $\mathrm{cl}_X\ A$. Analogously, the interior of A will be denoted by int A or $\mathrm{int}_X\ A$. The complement of a subset B of X will be denoted by $X\backslash B$.

As usual, $C(X)$ will denote the collection of all real-valued continuous functions on X. A zero-set in X is a set of the form $Z(f) = \{x \in X : f(x) = 0\}$ for some $f \in C(X)$. Complements of these are called cozero-sets. If $\Omega \subseteq C(X)$, then $Z(\Omega) = \{Z(f) : f \in \Omega\}$. By $Z(X)$ we mean the family of all zero-sets defined by $C(X)$, by $\mathrm{Bai}(X)$ we mean the Baire sets in X and by $\mathrm{Bo}(X)$ we mean the Borel sets in X.

Whenever Ω is a cofs on a space (set) X, we always assume that X is given the weak topology generated by Ω. Thus the sentence "Ω is a cofs on the space X" means that Ω is a cofs on X which generates the topology of X.

Every cofs Ω on a space X has associated a compactification $\beta(\Omega)$ and a realcompactification $\nu(\Omega)$ of the space X which, when Ω is the ring of all continuous functions on X, coincide with the Stone-Čech compactification and the Hewitt realcompactification of X respectively. See [8], [16], [7] and [15], for different constructions of these spaces. In [14] Lorch considers a Banach algebra of bounded continuous functions on a space X to obtain a compactification and a realcompactification of X which coincide with the ones above when the Banach algebra of continuous functions is Ω^*, the bounded functions of a cofs.

Given a cofs Ω on X we shall denote by $\Sigma(\Omega)$ the σ-algebra of sets generated by $Z(\Omega)$ and by $\mathrm{ba}(X, \Sigma(\Omega))$ the set of all real-valued bounded additive set functions defined on $\Sigma(\Omega)$. A function $\mu \in \mathrm{ba}(X, \Sigma(\Omega))$ is said to be regular if for every $A \in \Sigma(\Omega)$ and $\varepsilon > 0$ there is $F \in Z(\Omega)$ and $G \in \mathrm{coz}(\Omega)$ such that $F \subseteq A \subseteq G$ and the variation of μ over $G\backslash F$, $\mathrm{var}(\mu, G\backslash F)$, is less than ε. Let $\mathrm{rba}(X, \Sigma(\Omega))$ be the subset of $\mathrm{ba}(X, \Sigma(\Omega))$ consisting of all regular set functions.

By $B_1(\Omega)$ we mean the set of all pointwise limits of sequences of Ω, $B_2(\Omega) = B_1(B_1(\Omega))$ and in general, if α is an ordinal number, $\alpha > 0$, $B_\alpha(\Omega)$ is the family of all pointwise limits of sequences from $\cup\{B_\gamma(\Omega) : \gamma < \alpha\}$. Finally $B_{\omega_1}(\Omega)$ is denoted by $B(\Omega)$ (see [15] for discussion of Baire systems).

3. A Representation Theorem

In the sequel we shall give a representation theorem for positive linear functionals on a cofs Ω which separates points in X. Here, I is a positive linear functional on Ω means that I is linear and $I(f) \geq 0$ whenever $f \geq 0$ belongs to Ω.

We recall that X is provided with the weak topology induced by Ω. Thus X is a completely regular Hausdorff space. We also have that when $f \in \Omega$, there is a continuous extension of f, denoted $\hat{f}$, defined on $\nu(\Omega)$ (see [8], [16], [7] and [15]).

Theorem 1. *Let I be a positive linear functional on cofs Ω. Then*

(a) *There is a compact subset $K \subseteq \nu(\Omega)$ and a positive linear functional J on $C(K)$ such that*

$$I(f) = J(\hat{f}_{|K}) \text{ for every } f \in \Omega.$$

(b) *There is $\mu \in rba(X, \Sigma(\Omega))$ such that μ is countably additive and*

$$I(f) = \int_X f \, d\mu \text{ for every } f \in \Omega.$$

Proof. First we shall prove that for all $f \in \Omega^+$ (Ω^+ denotes the set of positive elements of Ω) there is $n \in \mathbb{N}$ such that $I(f \vee n) = I(n)$.

Let us suppose there is $f \in \Omega^+$ such that $I(f \vee n) > I(n)$ for all $n \in \mathbb{N}$. Then we define $g_n = (f \vee n) - n$ and we have that $I(g_n) = I(f \vee n) - I(n) > 0$. Let $g = \sum_{n=1}^{\infty} g_{n/I(g_n)}$ and let us see that $g \in \Omega$ (notice that g is well defined, as $g_n(x) = 0$ for $n \geq f(x)$).

Let $D_n = f^{-1}([n+1, \infty))$ and $C_n = f^{-1}((n, \infty))$ for all $n \in \mathbb{N}$. We have that coz $g_n = C_n$ and $C_n \supseteq D_n \supseteq C_{n+1}$ for all $n \in \mathbb{N}$; on the other hand $\cap\{C_n : n \in \mathbb{N}\} = \phi$. If we denote $g_{n/I(g_n)}$ by f_n then, for every open set V in $\mathbb{R}$, it is true that $g^{-1}(V) = \cup\{(f_1 + f_2 + \cdots + f_n)^{-1}(V) \cap (X \backslash D_n) : n \in \mathbb{N}\}$. Hence $g^{-1}(V) \in$ coz (Ω) and by ([9], Th. VIII) this means that $g \in \Omega$. Since $g \geq \sum_{n=1}^{p} f_n$ for all $p \in \mathbb{N}$, $I(g) \geq I(\sum_{n=1}^{p} f_n) = p$ for every $p \in \mathbb{N}$; that is a contradiction.

In the sequel we can suppose that I is a positive linear functional on Ω such that $I(1) = 1$.

Let us consider the compactification $\beta(\Omega)$ defined by Ω. We know (see [15], Th. 5.6) that for every $f \in \Omega^*$ there is a continuous extension of f, denoted $\dot{f}$, defined on $\beta(\Omega)$. The correspondence $f \to \dot{f}$ between Ω^* and $C(\beta(\Omega))$ defines an isomorphism which permits us to identify Ω^* with $C(\beta(\Omega))$. In this way, we can assume that I is a positive linear functional on $C(\beta(\Omega))$, by setting $I(\dot{f}) = I(f)$ for all $f \in \Omega^*$. Let us see that there is a smallest compact set $K \subseteq \beta(\Omega)$ such that $I(f) = 0$ when $\dot{f}$ is zero on K.

Let $\mathcal{F}$ be the collection of all compact subsets H of $\beta(\Omega)$ such that whenever $f \in \Omega^*$ and $\dot{f} = 0$ on H is follows that $I(f) = 0$. We claim that $\mathcal{F}$ has the finite intersection property.

In fact, if H_1 and H_2 belong to $\mathcal{F}$ and $H_1 \cap H_2 = \phi$, then there is a finite partition of unity in $C(\beta(\Omega))$ subordinated to the cover $\{\beta(\Omega) \backslash H_i : 1 \leq i \leq 2\}$. Clearly I is zero on the functions which belong to the partition of unity and so I is zero on $C(\beta(\Omega))$. This contradiction shows that $\mathcal{F}$ has the finite intersection property. Therefore $\cap\{H : H \in \mathcal{F}\} \neq \phi$.

Let $K = \cap\{H : H \in \mathcal{F}\}$, and let us see that $K \in \mathcal{F}$. Suppose $f \in \Omega^*$ such that $\dot{f} = 0$ on K. We define $A_n = \{p \in \beta(\Omega) : |\dot{f}(p)| \geq 1/n\}$. Since $A_n \cap K = \phi$, there is $H \in \mathcal{F}$ with $A_n \cap H = \phi$. Take $g_n \in \Omega$ such that $0 \leq g_n \leq 1$, $\dot{g}_n = 0$ on H and $\dot{g}_n = 1$ on A_n and let $f_n = f \cdot g_n$. The sequence $\{f_n\}$ converges uniformly to f and $I(f_n) = 0$ for all $n \in \mathbb{N}$. Since I is a positive functional, it is continuous with respect to the topology of the uniform convergence. Thus $I(f) = 0$, and $K \in \mathcal{F}$.

Finally, let us prove that K is included in $\nu(\Omega)$.

If $p \in K\backslash\nu(\Omega)$, then there is $f \in \Omega^*$ such that $\lim_{\delta\in D} f(x_\delta) = +\infty$ for every net in X, $\{x_\delta\}_{\delta\in D}$, which converges to p (see [8], Prop. 2.5). Let n be a natural number such that $I(f \vee n) = I(n)$; the function $g = (f \vee n) - n$ is such that $I(g) = 0$ and may be extended to a continuous function $\dot{g} : \beta(\Omega) \to [0, +\infty]$ (see [15], Th. 5.6). Let $H = \{q \in \beta(\Omega) : \dot{g}(q) \leq 1\} \cap K$ then $H \subseteq K$ and $H \neq K$. Take $h \in \Omega^*$ with $\dot{h} = 0$ on H and let us see that $I(h) = 0$; there is no loss of generality by supposing that $0 \leq h \leq 1$. We know that there is a function in Ω^*, r, with $0 \leq r \leq 1$, $\dot{r} = 0$ on $Z([\dot{g} - \frac{1}{2}] \wedge 0)$ and $\dot{r} = 1$ on $Z([\dot{g} - \frac{1}{3} \vee 0)$. Then $h \leq h \cdot r + 3g$ and $\dot{h} \cdot \dot{r} = 0$ on K, i.e., $I(h) \leq I(h \cdot r) + 3I(g) = 0$. This contradicts the property that the compact set K is the smallest compact subset of $\beta(\Omega)$ with the property above. Thus $K \subseteq \nu(\Omega)$.

Since K is compact, it is a C-embedded subset of $\beta(\Omega)$. On the other hand, $Z(\beta(\Omega)) \cap \nu(\Omega) = \{\mathrm{cl}_{\nu(\Omega)}D : D \in Z(\Omega)\}$ (see [2], Cor. 3.4). Thus for every $Z \in Z(K)$, there is $D \in Z(\Omega)$ with $Z = (\mathrm{cl}_{\nu(\Omega)}D) \cap K$.

Let us denote by $\hat{\Omega}$ the collection of all continuous functions on $\nu(\Omega)$ which are continuous extensions of functions in Ω. Then $\hat{\Omega}$ is a cofs on $\nu(\Omega)$ and $Z(\hat{\Omega}) = Z(\beta(\Omega)) \cap \nu(\Omega)$. Hence $Z(\hat{\Omega}) \cap K = Z(K)$ and $\Sigma(\hat{\Omega}) \cap K = \mathrm{Bai}(K)$.

Now we can prove part (a) of the Theorem. Let us define a positive linear functional on $C(K)$ as follows. For each $h \in C(K)$, let $\bar{h}$ denote any continuous extension of h to the space $\beta(\Omega)$ and state $J(h) = I(\bar{h}_{|X})$. J is a positive linear functional on $C(K)$ and it is well defined by the election of K. This proves part (a).

In order to prove part (b), note that I is a positive linear functional on Ω^* such that, in Lorch's terminology, the I-measure of each Baire set in $\beta(\Omega)\backslash\nu(\Omega)$ is zero; this is because $K \subseteq \nu(\Omega)$. Hence we can apply ([14], Th. 13) and we deduce that I is a Daniell integral on Ω^*, i.e., there is a bounded, countably additive set function μ on $\Sigma(\Omega)$ such that every function in Ω^* is μ-integrable and besides $I(f) = \int_X f \, d\mu$ for every $f \in \Omega^*$.

Let us see that every function in Ω is μ-integrable. Take $f \in \Omega^+$, then the sequence $\{f \wedge n\}_{n=1}^{\infty}$ is monotone-increasing and converges pointwise to f. Since $\int_X (f \wedge n) \, d\mu = I(f \wedge n) \leq I(f)$ for every $n \in \mathbb{N}$. Lebesgue's Monotone Convergence Theorem shows that f is μ-integrable and $\int_X f \, d\mu \leq I(f)$. On the other hand, we know that there is $m \in \mathbb{N}$ such that $I(f \vee m) = I(m)$; as

$f = (f \vee m) + (f \wedge m) - m$, we have that $I(f) = I(f \vee m) + I(f \wedge m) - I(m) = I(f \wedge m) = \int_X (f \wedge m)\, d\mu \leq \int_X f\, d\mu$, i.e., $I(f) = \int_X f\, d\mu$ for every $f \in \Omega^+$. Since every f in Ω may be written as the difference of two positive functions, we have proved that every function in Ω is μ-integrable and also $I(f) = \int_X f\, d\mu$ for every $f \in \Omega$. Now it is proved analogously to the proof of Theorem 14.2 of [5] that μ is a regular measure on $\Sigma(\Omega)$. This proves part (b) of the Theorem.

4. Some Applications

In the sequel we shall make use of the term P-space, that is, a space in which every G_δ (countable intersections of open sets) is open.

Proposition 2. *Let Ω be a cofs on a space X and suppose that $\nu(\Omega)$ is a P-space. Then every positive linear functional on Ω is the sum of a finite number of Riesz homomorphisms.*

Proof. Let I be a positive linear functional on Ω. Then there is a compact subset K of $\nu(\Omega)$ and a positive linear functional J on $C(K)$ such that $I(f) = J(\hat{f}_{|K})$ for all $f \in \Omega$. Since $\nu(\Omega)$ is a P-space we know that K is a finite subset of $\nu(\Omega)$. Hence there are $\{x_1, x_2, \ldots, x_n\} \subseteq \nu(\Omega)$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq \mathbb{R}$ such that $I(f) = J(\hat{f}_{|K}) = \sum_{j=1}^{n} \lambda_j \cdot \hat{f}(x_j)$ for all $f \in \Omega$, i.e., I is the sum of a finite number of Riesz homomorphisms.

Remark. Suppose that Φ is a vector lattice of functions which contains the constants, and let us denote by Ω the smallest cofs continuing Φ. We know that $B_\alpha(\Phi) = B_\alpha(\Omega)$ for every α such that $1 \leq \alpha \leq \omega_1$ (see [15], Th. 3.5). On the other hand $\nu(B_\alpha(\Omega)) = \nu(\Omega)_p$ for all α with $1 \leq \alpha \leq \omega_1$ (here $\nu(\Omega)_p$ denotes the same set $\nu(\Omega)$ endowed with the P-topology associated, i.e., the topology for which the collection of G_δ-subsets of $\nu(\Omega)$ form an open base). Thus $\nu(B_\alpha(\Omega))$ is a P-space for all α with $1 \leq \alpha \leq \omega_1$ (see [11], Th. 2.4.). Hence the Proposition above is a variation of Theorem 1 of [18] and Theorem 2 of [3].

Let Ω be a cofs on a space X, we can identify Ω with a subalgebra of $C(\nu(\Omega))$ by the embedding $f \longrightarrow \hat{f}$. In many cases the range of the embedding is different from $C(\nu(\Omega))$ and it is an unsolved problem to give a general method in order to generate the algebra $C(\nu(\Omega))$ internally from the cofs Ω (see [10], Problem 1 which is closely related). In the sequel we develop a formal method for obtaining $C(\nu(\Omega))$ from Ω by applying the ideas above. First we shall introduce some definitions.

Given a cofs Ω, consider the order dual space Ω'' (see [4] for definitions and notation) and let $(\Omega'')^+$ denote the positive elements of Ω''. It is known that every element of Ω'' is the difference of two elements of $(\Omega'')^+$ (see [4], 16B). We can suppose that Ω is included in the Dedekind complete Riesz space

$(\Omega'')''$ via the canonical evaluation map (see [4], Prop. 31C). We denote by $\Omega^{\uparrow} \cap \Omega^{\downarrow}$ the space of all $G \in (\Omega'')''$ such that $G = \sup\{f : f \in \Omega, f \leq G\}$ and $G = \inf\{g : g \in \Omega, g \geq G\}$ (this in the order of $(\Omega'')''$).

Theorem 3. *Let Ω be a cofs on a space X. Then $C(\nu(\Omega))$ is lattice and vector isomorphic to $\Omega^{\uparrow} \cap \Omega^{\downarrow}$.*

Proof. Let G be an element of $\Omega^{\uparrow} \cap \Omega^{\downarrow}$, by ([4], Th. 16D(b)) we know that for every $I \in (\Omega'')^+$ we have $G(I) = \sup\{I(f) : f \in \Omega, f \leq G\} = \inf\{I(g) : g \in \Omega, g \geq G\}$. Every point $p \in \nu(\Omega)$ defines an element of $(\Omega'')^+$, that we shall denote by δ_p, by defining $\delta_p(f) = \hat{f}(p)$. Hence we can define $\bar{G} : \nu(\Omega) \to \mathbb{R}$ by $\bar{G}(p) = G(\delta_p)$. From the equalities above we deduce that $\bar{G}(p) = \sup\{\hat{f}(p) : f \in \Omega, f \leq G\} = \inf\{\hat{g}(p) : g \in \Omega, g \geq G\}$. Therefore $\bar{G}$ is a continuous map on $\nu(\Omega)$.

Conversely, let F be a continuous map on $\nu(\Omega)$. We can suppose that F belongs to $(C(\nu(\Omega))'')''$ via the canonical evaluation map.

On the other hand, by Theorem 1, part (a), every positive linear functional defined on Ω (resp. $C(\nu(\Omega))$ is localized on a compact subset of $\nu(\Omega)$ (resp. $\nu(C(\nu(\Omega))))$. Since $\nu(\Omega)$ is a realcompact space we have that $\nu(C(\nu(\Omega))) = \nu(\Omega)$, i.e., Ω and $C(\nu(\Omega))$ have the same realcompact spaces associated. Thus every positive linear functional on Ω is also defined on $C(\nu(\Omega))$ and vice versa. Therefore it follows that $\Omega'' = C(\nu(\Omega))''$, i.e., we can assume that F belongs to $(\Omega'')''$.

By Theorem 1, part (a), every $I \in (\Omega'')^+$ has associated a compact subset of $\nu(\Omega)$, K_I, and a positive linear functional on $C(K_I), J$, such that $I(f) = J(\hat{f}_{|K_I})$. Applying ([13], Th. 9), we get a regular Baire measure on K_I, μ, such that $J(h) = \int_{K_I} h \, d\mu$ for every $h \in C(K_I)$. Thus $I(f) = \int_{K_I} \hat{f}_{|K_I} \, d\mu$ for every $f \in \Omega$. Since $\Omega'' = C(\nu(\Omega))''$ we also have that $I \in C(\nu(\Omega))''$ and that $I(F) = \int_{K_I} F_{|K_I} \, d\mu$.

Let us see that $F = \sup\{f : f \in \Omega, f \leq F\}$ in the natural order of $(\Omega'')''$, i.e., we must prove that $I(F) = \sup\{(I(f) : f \in \Omega, f \leq F\}$ for every $I \in (\Omega'')^+$.

Suppose that I is in $(\Omega'')^+$ and let K_I be the compact set associated with I as before. For every $x \in K_I$ there is $U_x \in$ coz $\hat{\Omega}$ such that osc $(F, U_x) < \varepsilon$. Since $Z(\hat{\Omega})$ is a normal base on $\nu(\Omega)$ (see [16], Th. 4.3), there is $D_x \in Z(\hat{\Omega})$ such that $x \in$ int $D_x \subseteq U_x$. The collection $\{\text{int } D_x : x \in K_I\}$ is an open cover of K_I. Let $\{\text{int } D_1, \ldots, \text{int } D_n\}$ be a finite subcover of K_I and consider also the finite open cover $\{U_1, \ldots, U_n\}$. For every U_i there is $\alpha_i \in \hat{\Omega}$ such that $0 \leq \alpha_i \leq 1$ and $Z(\alpha_i) = \nu(\Omega) \backslash U_i$. Consider also $\beta \in \hat{\Omega}$ with $0 \leq \beta \leq 1$ and $Z(\beta) = D_1 \cup \cdots \cup D_n$. Set $\phi = \sum_{i=1}^{n} (\alpha_i \vee \beta)$ and $\phi_i = \frac{\alpha_i}{\phi}$. Then $\{\phi_i : 1 \leq i \leq n\}$ is a partition of unity in K_I subordinated to $\{U_1, \ldots, U_n\}$, $\phi_i \in \hat{\Omega}$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} \phi_i \leq 1$ in $\nu(\Omega)$.

Take a fixed point x_i of U_i for each $i \in \{1, \ldots, n\}$ and define the following function on X, $f = \sum_{i=1}^{n}(F(x_i)-\varepsilon)\cdot\phi_i$. Then $f \in \hat{\Omega}$, $f \leq F$ and $|F(x)-f(x)| < \varepsilon$ for every $x \in K_I$. Hence we can approximate F on K_I by functions in $\hat{\Omega}$ which are bounded above by F. By Lebesgue's Monotone Convergence Theorem, we get $I(F) = \int_{K_I} F_{|K_I}\, d\mu = \sup\{\int_{K_I} \hat{f}_{|K_I} d\mu : f \in \Omega,\ f \leq F\} = \sup\{I(f) : f \in \Omega, f \leq F\}$. Analogously it is proved that $I(F) = \inf\{(I(g) : g \in \Omega, g \geq F\}$. This proves that $F \in \Omega^{\uparrow} \cap \Omega^{\downarrow}$.

Theorem 4. *Let X and Y be two spaces and let Ω be a cofs on X. If $T : \Omega \to C(Y)$ is a positive linear operator then it may be extended to a positive linear operator $\hat{T} : C(\nu(\Omega)) \to C(Y)$.*

Proof. Let $y \in Y$ and let δ_y denote the evaluation map on y. The composition $\delta_y \circ T$ is a positive linear functional on Ω which, by the Theorem above, may be extended to a positive linear functional, $(\delta_y \circ T)^{\frown}$, on $C(\nu(\Omega))$ as follows:

$$(*)\ (\delta_y \circ T)^{\frown}(f) = \sup\{(\delta_y \circ T)(g) : g \in \Omega, g \leq f\} = \inf\{(\delta_y \circ T)(h) : h \in \Omega, h \geq f\}.$$

This enables us to define a map $\hat{T}(f)$ on Y, for every $f \in C(\nu(\Omega))$, by $\hat{T}(f)(y) = (\delta_y \circ T)^{\frown}(f)$ for all $y \in Y$ and, from the equalities in $(*)$, we deduce that $\hat{T}(f) \in C(Y)$.

Remark. If Φ is a linear lattice of functions on a space X, we have that $\nu(B_1(\Phi)) = \nu(B_\alpha(\Phi))$ for all α such that $1 \leq \alpha \leq \omega_1$ (see [11], Th. 2.4). Thus $B(\Phi) \subseteq C(\nu(B_1(\Phi)))$. Let us suppose that $T : B_1(\Phi) \to C(Y)$ is a positive linear operator. By applying the Theorem above T may be extended to $C(\nu(B_1(\Phi)))$ and, since $B(\Phi) \subseteq C(\nu(B_1(\Phi)))$, T may be extended to a positive linear operator $\hat{T} : B(\Phi) \to C(Y)$. Taking this into account we see that the Theorem above contains Corollary 3 of [18], Theorem 9 of [19] and Theorem 3 of [3].

Corollary. *Let Φ be a linear lattice of functions containing the constants and let $T : B_1(\Phi) \to C(Y)$ be a positive linear operator. If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $B_1(\Phi)$ which converges pointwise to a function f, then the sequence $\{T(f_n)\}_{n=1}^{\infty}$ converges pointwise to a function in $C(Y)$.*

Proof. By the remark above we can extend T to a positive linear operator $\hat{T} : B(\Phi) \to C(Y)$. Since $f \in B(\Phi)$, we have that $\hat{T}(f) \in C(Y)$. Let us see that $\{T(f_n)\}$ converges pointwise to $\hat{T}(f)$.

Take $y \in Y$ and consider $\delta_y \circ \hat{T}$ which is a positive linear functional on $B(\Phi)$. By Proposition 2, there is $\{x_1, \ldots, x_n\} \subseteq \nu(B(\Phi))$ and $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{R}$ such that $\delta_y \circ \hat{T} = \sum_{i=1}^{n} \lambda_i \cdot \delta_{x_i}$. Thus, in order to prove that $\{T(f_n)\}$ converges pointwise to $\hat{T}(f)$, we only need to prove that $\{f_n\}$ converges pointwise to f in $\nu(B(\Phi))$.

Let us suppose that there is $x \in \nu(B(\Phi))$ and $\varepsilon_0 > 0$ such that $|f_{n_j}(x) - f(x)| \geq \varepsilon_0$ for every n_j belonging to a sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$. Let $U_n = \{y : y \in \nu(B(\Phi)),\ |f_n(y) - f(y)| \geq \varepsilon_0\}$. Then $x \in \cap\{U_{n_j} : j \in \mathbb{N}\}$ which is a zero-set in $\nu(B(\Phi))$. Hence $\cap\{U_{n_j} \cap X) : j \in \mathbb{N}\} \neq \phi$, which is a contradiction.

Remark. This result contains Theorem 4 of [17]. Also, it is easily checked that if T is a positive linear map defined on a cofs and with values on a space of functions, then T satisfies Lebesgue's Monotone Convergence Theorem and Lebesgue's Dominated Convergence Theorem. This means that if $\{f_n\}$ is a sequence pointwise convergent to f and the sequence satisfies the further conditions of any of the two theorems mentioned previously, the sequence $\{T(f_n)\}$ converges pointwise to $T(f)$. For example, let us suppose $T : \Omega \to \mathbb{R}^Y$, $\{f_n : n \in \mathbb{N}\}$ a sequence in Ω which converges pointwise to f, and $g \in \Omega$ with $|f_n| \leq g$ for all $n \in \mathbb{N}$. For every $y \in Y$, $\delta_y \circ T$ is a positive functional on Ω which, by Theorem 1, satisfies Lebesgue's Dominated Convergence Theorem. Thus f is $\delta_y \circ T$-integrable and $\{(\delta_y \circ T)(f_n) : n \in \mathbb{N}\}$ converges to $\delta_y \circ T)(f)$. Since $(\delta_y \circ T)(f_n) = T(f_n)(y)$, if we define $T(f)(y) = (\delta_y \circ T)(f)$, we obtain that $\{T(f_n)\}$ converges pointwise to $T(f)$.

References

[1] A.D. Alexandroff, *Additive set-functions in abstract spaces*, Mat. Sb. (8) 50 (1940), 307-342; (9) 51 (1941), 563-621.

[2] R.A. Alo and M. Weir, *Realcompactness and Wallman realcompactifications*, Port. Math. 34 (1975), 33-43.

[3] F.S. Cater, *Rings of Baire Functions on Realcompact Spaces*, Real Analysis Exchange, 11 (1985-86), 323-346.

[4] D.H. Fremlin, *Topological Riesz Spaces and Measure Theory*, Cambridge University Press, 1974.

[5] R.J. Gardner and W.F. Pfeffer, *Borel Measures*, Handbook of Set-Theoretic Topology, North-Holland, 1984, 347-422.

[6] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand Co., Princeton, 1960.

[7] H. Gordon, *Rings of functions determined by zero-sets*, Pacific J. Math. 36 (1971), 133-157.

[8] A.W. Hager, *On inverse-closed subalgebras of $C(X)$*, Proc. London Math. Soc. 19 (1968), 233-257.

[9] F. Hausdorff, *Set Theory*, Chelsea, New York, 1957.

[10] M.E. Henriksen, *Unsolved Problems on Algebraic Aspects of $C(X)$*, pp. 90-123 of Rings of Continuous Functions, edited by Ch. E. Aull, Lecture Notes in Pure and Applied Math., Vol. 95, Dekker 1985.

[11] S. Hernandez, *Algebras de funciones de Baire*, Rev. Real Acad. de Ciencias Exactas Físicas y Naturales 81 (1987), 203-208.

[12] E. Hewitt, *Linear Functionals on Spaces of Continuous Functions*, Fund. Math. 37 (1950), 161-189.

[13] S. Kakutani, *Concrete Representations of Abstract M-spaces*, Anals of Math. 42 (1941), 994-1024.

[14] E.R. Lorch, *Compactifications, Baire functions, and Daniell integration*, Acta Sci. Math. (Zseged) 24 (1963), 204-218.

[15] R.D. Mauldin, *Baire functions, Borel sets and ordinary functions systems*, Advances Math. vol. 12 (1974), 51-59.

[16] A.K. Steiner and E.F. Steiner, *Nest generated intersection rings in Tychonoff spaces*, Trans. Amer. Math. Soc. 148 (1970), 589-601.

[17] C.T. Tucker, *Representation of Baire functions as continuous functions*, Fund. Math. 101 (1981), 181-188.

[18] C.T. Tucker, *Positive operators on spaces of Baire functions*, Illinois J. Math. 25 (1981), 295-301.

[19] C.T. Tucker, *Pointwise and order convergence for spaces of continuous functions and spaces of Baire functions*, Czech. Math. J. 34 (109) (1984), 562-569.