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Packing Measures in Different Bases and Probability Theory

Patrick Billingsley [1] defines the Hausdorff measure in bases other than base two. These "Hausdorff measures" are equivalent to the standard Hausdorff measure in that both are zero, finite, or infinite on the same set. The "probability" in the author's paper was omitted from the author's talk.

The author attempted to define the packing measure in different bases so that the definition would be the same as Tricot and Taylor's work with semi–dyadic numbers and so the packing measure in different bases would be equivalent to the standard packing measure.

The definition of the Base Measures is as follows:

Definition 1 *(Base Measures) Let E be any set in $\mathbb{R}^m$ and let t be any natural number such that $t \geq 2$. Let l_i and k_i, $i = 1 \ldots, m$ be any integers such that $t^2 \geq k_i - l_i \geq 3$ and $k_1 - l_1 = k_2 - l_2 = \ldots = k_m - l_m$. Let n be any natural number and let $\delta(x)$ be any positive real function defined on $\mathbb{R}^m$. Let $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$ and $\limsup_{r \to 0} \frac{h(2r)}{h(r)} = h^* < \infty$. Let $I = \prod_{i=1}^{m} \left[\frac{l_i}{t^n}, \frac{k_i}{t^n}\right] \subset \mathbb{R}^m$ with the diameter of I, $|I|$, satisfying $\frac{3\sqrt{m}}{t^n} \leq |I| \leq \frac{t\sqrt{m}}{t^{n-1}}$. Also, a point $x \in E \cap \left\{\prod_{i=1}^{m} \left[\frac{l_i+1}{t^n}, \frac{k_i-1}{t^n}\right]\right\}$ and $I \subset B(x, \delta(x))$. Then $H_t(E) = \sup\{\sum_i h(|I_i|) : I_i$ is any non–overlapping sequence of intervals in $\mathbb{R}^m$ defined as above for a given positive $\delta(x)\}$. The base measure is $h_t(E) = \inf\{H_t(E) : \delta(x)$ is any positive real function$\}$.*

The following observation shows that if you take a specific packing by balls, that a cube, defined in the Definition can be placed inside the ball. This is needed for Theorem 1.

Observation 1 *Let $B(x, r)$ be any ball in $\mathbb{R}^m$ and let t be any natural number such that $t \geq 2$. Then, there exists a natural number n such that $\frac{1}{t^{n+1}} < 2r \leq \frac{1}{t^n}$. There exists a natural number u such that $\frac{\sqrt{m}}{t^u} < \frac{1}{2}$ and the interval $I = \prod_{i=1}^{m} \left[\frac{l_i}{t^{n+u+1}}, \frac{k_i}{t^{n+u+1}}\right]$ has the following properties*

(i) $I \subset B(x, r)$

(ii) $x \in \prod_{i=1}^{m} \left[\frac{l_i+1}{t^{n+u+1}}, \frac{k_i-1}{t^{n+u+1}}\right]$

(iii) $\frac{3\sqrt{m}}{t^{n+u+1}} \leq |I| \leq \frac{t\sqrt{m}}{t^{n+u}}$

In the theorem that follows, $h_p(E)$ is the standard packing measure.

Theorem 1 *Let t be any natural number and let E be any set in $\mathbb{R}^m$. Let $q(t)$ be a natural number such that $\frac{t^{u+1}}{3\sqrt{m}} \leq 2^{q(t)}$. Then, $h_p(E) \leq \left[1 + (h^*)^{q(t)}\right] h_t(E)$.*

From the definition of the base measures, it is clear that a ball can be placed inside a cube. Therefore, Theorem 2 follows:

Theorem 2 *Let t be a natural number and let E be any set in $\mathbb{R}^m$. Let $q(t)$ be any natural number such that $\frac{t^2}{2} \leq 2^{q(t)}$. Then, $h_t(E) \leq \left[1 + (h^*)^{q(t)}\right] h_p(E)$.*

If $w(E, n)$ is defined to be the number of t^{-n} intervals that intersect E, the author was attempting to show that $\alpha = \inf\{\beta : \sum_{n=1}^{\infty} w(E, n)(t^{-n})^{\beta}$ converges$\}$ was the packing measure dimension of E. However, it turns out to be the dimension of the packing premeasure.

Theorem 3 *Let $E \subset [0, 1]$, and let*

$$\gamma = \limsup_{n \to \infty} \frac{\log w(E, n)}{\log t^n}$$

and

$$\alpha = \inf\{\beta : \sum_{n=1}^{\infty} w(E, n)(t^{-n})^{\beta} \textit{ converges}\}.$$

Then $\gamma = \alpha$.

References

[1] P. Billingsley, *Hausdorff dimension in probability theory*, Illinois J. Math. **4** (1960), 187–209.