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Weighted Inequalities in Function Spaces

The problem of characterizing non-negative locally integrable (weight) functions u and v on $\mathbb{R}^+$, for which Hardy's inequality

$$\left\{\int_0^\infty u(x)\left[\int_0^x f(t)dt\right]^q dx\right\}^{\frac{1}{q}} \le C\left\{\int_0^\infty v(x)f(x)^p dx\right\}^{\frac{1}{p}}, 0 < p, q < \infty, \quad (1)$$

holds for all non-negative $f \in L_v^p$ has been completely solved. The formulation for the discrete version of this result in the index range $1 < p, q < \infty$ is

Theorem 1 ([1],[4]) *Suppose $1 < p, q < \infty$ and $\{u_k\}_{k\in\mathbb{N}}$, $\{v_k\}_{k\in\mathbb{N}}$ are sequences such that $u_k \ge 0$, $v_k > 0$, $k \in \mathbb{N}$. Then there exists a constant $B > 0$, such that*

$$\left\{\sum_{n\in\mathbb{N}} u_n \left[\sum_{k=1}^n a_k\right]^q\right\}^{\frac{1}{q}} \le B\left\{\sum_{n\in\mathbb{N}} v_n a_n^p\right\}^{\frac{1}{p}} \quad (2)$$

holds for all non-negative sequences $\{a_k\} \in \ell^p_{\{v_n\}}$, if and only if

(i) in case $1 < p \le q < \infty$

$$B_1 = \sup_{m\in\mathbb{N}}\left\{\sum_{n=m}^\infty u_n\right\}^{\frac{1}{q}}\left\{\sum_{n=1}^m v_n^{1-p'}\right\}^{\frac{1}{p'}} < \infty$$

(ii) in case $1 < q < p < \infty$

$$B_2 = \left\{\sum_{m\in\mathbb{N}}\left[\sum_{n=m}^\infty u_n\right]^{\frac{r}{q}}\left[\sum_{n=1}^m v_n^{1-p'}\right]^{\frac{r}{q'}} v_m^{1-p'}\right\}^{\frac{1}{r}} < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and p', q' are the conjugate indices of p and q, respectively.

In this talk we discuss weight characterizations of inequalities of the form (1) and (2), where the weighted Lebesgue spaces are replaced by more general function spaces. Specifically, we consider weighted amalgams $\ell^q(L^p_w)$, $1 < p, q < \infty$, which consist of functions locally in L^p_w and whose integrals over $[n, n+1]$ form an ℓ^q sequence. Under the norm

$$\|f\|_{p,w,q} = \left\{ \sum_{n \in \mathbb{N}} \left[\int_{n-1}^{n} w(x)|f(x)|^p dx \right]^{\frac{q}{p}} \right\},$$

$\ell^q(L^p_w)$ becomes a Banach space.

A generalization of (1) and (2) in a different direction is obtained by replacing the weighted Lebesgue spaces by weighted Orlicz spaces. We discuss, in particular, under what conditions the modular inequality

$$Q^{-1}\left\{ \sum_{n \in \mathbb{N}} Q\left[u_n^1 \sum_{k=1}^{n} a_k \right] u_n^0 \right\} \leq A P^{-1} \left\{ \sum_{n \in \mathbb{N}} P\left(a_n v_n^1\right) v_n^o \right\} \qquad (3)$$

holds for all non-negative sequences $\{a_n\}_{n \in \mathbb{N}}$ and certain Young's functions P and Q. In fact, conditions on the weight sequences $\{u_n^j\}_{n \in \mathbb{N}}$, $\{v_n^j\}_{n \in \mathbb{N}}$, $j = 0, 1$ are given which are equivalent to (3). Of course, the choice $Q(x) = x^q$, $P(x) = x^p$, $1 < p \leq q < \infty$, $u_n^1 = v_n^1 = 1$ for $n \in \mathbb{N}$ in (3) yields Theorem 1(i).

In higher dimensions, applications of the corresponding continuous form of the inequality provide weighted Friedrich type inequalities.

References

[1] G. Bennett. Some elementary inequalities III. *Quart. J. Math., Oxford Ser. (2)*, 42 (1991), 149–174.

[2] C. Carton–Lebrun, H. P. Heinig, S. C. Hofmann. Integral operators on weighted amalgams (Preprint).

[3] H. P. Heinig, L. Maligranda. Interpolation with weights in Orlicz spaces. *Boll. della Unione Math. Ital.* (to appear).

[4] G. Sinnamon. Spaces defined by their level function and their dual (Preprint).