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Translates of a Set Which Meet It in a Set of Positive Measure

It is well known that the Cantor ternary set C satisfies $C + C = [0, 2]$. One can observe this by considering the set $C \times C$ and noting that this set meets each line $x + y = k$ when $k \in [0, 2]$. It is also easy to observe from $C \times C$ that lines $x + y = k$ which intersect $C \times C$ in a set of positive s'–measure ($s' = \log 2/\log 3$) are those which pass through the corners of squares in the construction of $C \times C$; that is, points (x, y) where x and y are endpoints of intervals contiguous to C. This implies that there are exactly countably many numbers a so that $(C+a) \cap C$ has positive s'–measure. This yields some curious open questions regarding s–sets (measurable sets of non–zero finite s–measure): Given a compact s–set E in $\mathbb{R}^n$ with $s < n$, how large can the s–measure of $\{t : s\text{-m}((E + t) \cap E) > 0\}$ be? Perhaps it can have positive s–measure? Perhaps it can be no larger in dimension that $[[s]]$? If $E \subset \mathbb{R}^n$ is an s–set where $s < n$ is not a whole number, can $E + E$ be an s–set?

It is shown in the paper on which this talk is based that any singular, σ–finite, Borel regular measure m_a whose support is E (with $m(E) = 0$ in $\mathbb{R}^n$) satisfies $m(\{t : m_a((E + t) \cap E) > 0\}) = 0$. From this result it follows that, if E is an s–set or even a set of σ–finite s–measure in $\mathbb{R}^n$ with $s < n$, then $m(\{t : s\text{-}m((E + t) \cap E) > 0\}) = 0$. This fact is then used to show that each s–set in $\mathbb{R}^n$ with $s < n$ is a non–measurable set with respect to any of the approximating measures $s\text{-}m_\delta$ for any $\delta > 0$.