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The Problem of Integral–Geometric Uniqueness

This work springs from an ancient question asked by the late H. Steinhaus:

> If every line ℓ in $\mathbb{R}^2$ cuts each of two rectifiable plane arcs at the same number of points (0, a positive integer, or ∞), must the two arcs be equal?

The same, or almost the same, question may be asked concerning two rectifiable Borel sets instead of two rectifiable arcs.

For a fixed $E \subset \mathbb{R}^2$ and variable line ℓ the expression

$$N_E(\ell) = |E \cap \ell|_0 = \int_\ell \chi_E(z)|dz|_0$$

where integration is with respect to the *counting* measure on ℓ. I call this function the *Crofton function of* E. Let $|\cdot|_2$ be the integral–geometric measure in L, the space of all lines of $\mathbb{R}^2$. For $E \in Bor(\mathbb{R}^2)$, $|E|_2 < \infty$, it is known that $\int_L N_E|d\ell|_2 = 2|E|_2$; hence, $N_E < \infty$ a.e. on L. A *cross integral*, $\int N_{E|z}$, is the result of averaging N_E over the bundle of lines passing through $z \in \mathbb{R}^2$.

The cross integral of E equals the cross integral of the Besicovitch regular part since it vanishes for the irregular part. Hence, the question must be asked only about the regular parts of two sets.

Under an additional assumption that the sets under consideration have closed regular parts (which, since they are rectifiable simplies their nowhere density in the plane) the cross integral of E has an integral representation by an integral in linear measure over E, a fact which enables us to study its interior and boundary properties as a function of z in the domain which is the complement if E.

Two Borel rectifiable sets of the class under study whose cross integrals coincide over a subset essentially dense in $\mathbb{R}^2$ or, what is equivalent, whose Crofton functions coincide a.e. on L would have cross integrals coinciding a.e. in $\mathbb{R}^2$. Thus, the integral representations of the cross integral taken over one and then the other set would have to be a.e. equal in $\mathbb{R}^2$.

On the other hand, should the two sets be other than almost equal, the two integral representations would have to have both interior as well as boundary characteristics at the same points, which is impossible since such characteristics are incompatible with each other.

The apparent analogy between the Radon transform which assigns to an $\mathbb{R}^2$–integrable f its integral $I_f(\ell) = \int_\ell f|dz|$ and the Crofton function (assigning an integral to a characteristic function of E) might suggest using in this case similar analytic tools to those used to restore the source function f from its line integrals in the case of Radon transforms (*i.e.*, to invert the transform). This, however, turns out not to be possible and a variety of elementary set–theoretical and measure–theoretical techniques are used in the proof. Because of this lack of elegance of method, the inversion of the transform which thus has been achieved is somewhat awkward.

A number of further questions aimed at improving and sharpening these results are asked.