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Geometric origin and some properties of the arctangential heat equation

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We establish the geometric origin of the nonlinear heat equation with arctangential nonlinearity: $\partial_t D = \Delta(\arctan D)$ by deriving it, together and in duality with the mean curvature flow equation, from the minimal surface equation in Minkowski space-time, through a suitable quadratic change of time. After examining various properties of the arctangential heat equation (in particular through its optimal transport interpretation à la Otto and its relationship with the Born–Infeld theory of electromagnetism), we briefly discuss its possible use for image processing, once written in nonconservative form and properly discretized.

Introduction. The arctangential heat equation

$$\partial_t D = \Delta(\arctan D) \tag{1}$$

belongs to the class of degenerate nonlinear heat equations

$$\partial_t D = \Delta(\phi(D)),$$

(where ϕ is monotonic with derivative valued in $[0, +\infty]$), usually called "porous medium" (as $\phi(D) = D^m,\ m > 1$) or "fast diffusion" ($m < 1$) and sometimes related to geometry (such as $\phi(D) = \log D$, which corresponds to the Ricci flow in two space dimensions) [Brézis and Crandall 1979; Daskalopoulos and Kenig 2007; Topping and Yin 2017; Vázquez 2007]. The analysis of the arctangential heat equation from the usual PDE viewpoint (existence, uniqueness, regularity theory, ...) is not the point of the present paper. We rather show that the arctangential heat equation has a geometric origin and can be formally derived, together (and in duality) with the well known mean curvature flow for graphs, from the minimal surface equation in the Minkowski space of all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, with metric $-dt^2 + \delta_{ij} dx^i dx^j$. The minimal surface equation reads (see [Lindblad 2004], for example)

$$\partial_t\Big(\frac{\partial_t \phi}{R}\Big) = \partial_k\Big(\frac{\partial^k \phi}{R}\Big), \qquad R = \sqrt{1 - \partial_t \phi^2 + \partial^k \phi\, \partial_k \phi}, \tag{2}$$

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where

$$\phi = \phi(t, x) \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \quad \partial_k = \frac{\partial}{\partial x^k}, \quad \partial^k = \delta^{kj} \partial_j.$$

From this equation, properly expressed, in Section 1, as a "system of conservation laws with convex entropy" (in the sense of [Dafermos 2016]), we generate in Section 2, using the quadratic change of time method recently discussed in [Brenier and Duan 2018], two "dual" nonlinear parabolic equations: one is the well-known mean curvature flow (for graphs)

$$\partial_t \phi = \sqrt{1 + \partial_k \phi \partial^k \phi}\, \partial_i \left(\frac{\partial^i \phi}{\sqrt{1 + \partial_k \phi \partial^k \phi}} \right), \tag{3}$$

while the second one is precisely the arctangential heat equation (1). The arctangential heat equation seems widely ignored in the literature, but has, in our opinion, many interesting properties, discussed in Sections 3 and 4, on top of being "dual" to the mean curvature flow. First of all, we will compare, in Section 3, the arctangential heat equation, properly rescaled as

$$\partial_t D = \lambda \Delta(\arctan(D \lambda^{-1}))$$

with a large parameter $\lambda > 0$, to its formal limit as $\lambda \to +\infty$, namely the linear heat equation

$$\partial_t D = \Delta D,$$

both written à la [Otto 2001; Otto and Westdickenberg 2005], in optimal transport style (for which we refer to [Ambrosio et al. 2008; Otto and Westdickenberg 2005; Santambrogio 2015; Villani 2003]):

$$\partial_t D = \partial_i \big(D\, \partial^i (\mathcal{F}'(D)) \big).$$

In the linear case, $\mathcal{F}(D)$ is just the Boltzmann entropy function $D \log D - D$ (so that $\mathcal{F}'(D) = \log D$), in other words, the Legendre–Fenchel transform of the exponential function

$$\mathcal{F}(D) = D \log D - D = \sup_{u \in \mathbb{R}} uD - \exp u,$$

while, in the arctangential case, as will be seen in Section 3,

$$\mathcal{F}(D) = \sup_{u \le \log \lambda} uD - \lambda \arcsin(\lambda^{-1} \exp u),$$

is the Legendre–Fenchel transform of $u \to \lambda \arcsin(\lambda^{-1} \exp(u))$ (extended by $+\infty$ for $u > \log \lambda$), which can be seen as a "catastrophic" correction to the usual exponential function. (By catastrophic, we mean that this monotonic convex function reaches the value $\lambda\pi/2$ with infinite slope at $u = \log\lambda$ and, then, suddenly jumps to $+\infty$.)

Still in Section 3, we will briefly mention the "Chaplygin heat equation"

$$\partial_t D + \Delta(D^{-1}) = 0,$$

formally obtained in the opposite regime $\lambda \downarrow 0$. (This equation has been named "Chaplygin heat equation" because it can been interpreted as a "friction-dominated" version of the "Chaplygin gas", for which we refer to [Serre 2009].) We will also establish a connection between the arctangential heat equation and the nonlinear theory of Electromagnetism proposed by Max Born and Leopold Infeld [1934]. More precisely, in two space dimensions, both the arctangential heat equation and the mean curvature flow just describe *special* solutions, depending only on two space variables, of the *same* vector-valued diffusion equation in three space dimensions,

$$\partial_t D = \nabla \times \left(B\sqrt{1+D^2} - \frac{(D\cdot B)D}{\sqrt{1+D^2}} \right), \quad B = -\nabla \times \left(\frac{D}{\sqrt{1+D^2}} \right)$$

(written in traditional "nabla" notations, B and D being three-dimensional vectors, $\times$ denoting the vector product, $D\cdot B = D_k B^k$, $D^2 = D_k D^k$), which, itself, can be formally derived, again by quadratic change of time, from the Born and Infeld [1934] equations (for which we also refer to [Brenier 2004; Serre 2004]).

Finally, in Section 4, we discuss the nonconservative form of the arctangential heat equation:

$$\partial_t \psi = \cos(\pi\psi)^2 \Delta\psi \tag{4}$$

(where D is written as $\tan(\pi\psi)$). Properly discretized, this equation might be a valuable tool to treat black and white images (ψ denoting the level of gray), by sharply enhancing the level sets $\{\psi = k + \frac{1}{2}\}$ for $k \in \mathbb{Z}$ as t grows, as shown by several numerical computations in Section 4.

1. Reformulation of the minimal surface equation

It is crucial for our analysis to get a formulation of the minimal surface equation (2) in the framework of "systems of conservation laws with convex entropy", for which we refer to Dafermos' book [2016]. More precisely:

Theorem 1. *Let ϕ be a smooth solution of the minimal surface equation (2) and define*

$$D = \frac{\partial_t\phi}{\sqrt{1-\partial_t\phi^2 + \partial^k\phi\,\partial_k\phi}}, \quad B_i = \partial_i\phi, \quad P_i = \frac{-\partial_t\phi\,\partial_i\phi}{\sqrt{1-\partial_t\phi^2 + \partial^k\phi\,\partial_k\phi}}. \tag{5}$$

Then (D, B, P) *is a solution to the system of conservation laws*

$$\partial_t B_i + \partial_i\left(\frac{B_j P^j - D}{h}\right) = 0, \quad \partial_t D + \partial_j\left(\frac{D P^j - B^j}{h}\right) = 0, \tag{6}$$

$$\partial_t P^i + \partial_j\left(\frac{P^i P^j + B^i B^j}{h}\right) = \partial^i\left(\frac{1 + B_j B^j}{h}\right), \tag{7}$$

$$h(D, B, P) = \sqrt{1 + D^2 + B_j B^j + P_j P^j}, \tag{8}$$

which admits, for this strictly convex function h, *the extra conservation law*

$$\partial_t h + \partial_j\left(P^j - \frac{(DB^j + P^j) + B_k(B^k P^j - P^k B^j)}{h^2}\right) = 0. \tag{9}$$

Notice that the local in time existence of smooth solutions to the minimal surface equation (2) is a well known fact, while the global existence of smooth solutions for "small" (in a suitable sense) initial conditions for $d \geq 2$ is a much more refined result, obtained by Lindblad [2004].

Proof of Theorem 1. The proof follows a strategy similar to the one used for nonlinear Maxwell's equations, in particular for the Born–Infeld equations, in [Brenier 2004; Serre 2004]. In the first step, we get the Hamiltonian form of the minimal surface equations (2), which reads as a system of conservation laws for (D, B) (as defined in (5)), with an extra conservation law for $H(D, B) = \sqrt{(1 + B_k B^k)(1 + D^2)}$, which is a locally (but not globally) convex function of (D, B) about $(0, 0)$. The second step is a suitable augmentation of the Hamiltonian system in order to get a larger system of conservation laws, namely (6)–(7) for (D, B) *and* $P = -DB$. This new system enjoys an extra conservation law for the strictly convex function $h(D, B, P) = \sqrt{1 + D^2 + B_k B^k + P_k P^k}$ which is nothing but $H(D, B)$, written as a function of (D, B, P). The comprehensive proof of Theorem 1 can be found in the Appendix.

2. Recovery of the mean curvature flow and the arctangential heat equation from the minimal surface equation by quadratic change of time

Inspired by our recent work with X. Duan [Brenier and Duan 2018], we investigate the augmented system (6)–(8) under the quadratic change of time: $t \to \theta = t^2/2$. We consider two "dual" regimes of initial conditions at $t = 0$, well suited to this quadratic change of time, respectively $D(0, x) = 0$ and $B(0, x) = 0$ and, in both cases, $P(0, x) = 0$.

In the first regime, we assume $D(0, x) = 0$, $P(0, x) = 0$, i.e., in terms of the original field ϕ, the solution to (2), $\partial_t \phi(0, x) = 0$, by definition (5). We make the consistent ansatz

$$B(t,x)=\mathcal{B}(\theta,x),\quad D(t,x)=t\mathcal{D}(\theta,x),\quad P(t,x)=t\mathcal{P}(\theta,x),\quad \theta=t^2/2, \tag{10}$$

for some smooth fields $(\mathcal{B},\mathcal{D},\mathcal{P})$. In other words, we introduce

$$\mathcal{B}(\theta,x)=B(\sqrt{2\theta},x),\quad \mathcal{D}(\theta,x)=\frac{D(\sqrt{2\theta},x)}{\sqrt{2\theta}},\quad \mathcal{P}(\theta,x)=\frac{P(\sqrt{2\theta},x)}{\sqrt{2\theta}}.$$

In the second, "dual" regime, we assume $B(0,x)=0$, $P(0,x)=0$, which means $\partial_i\phi(0,x)=0$, in terms of ϕ, and, accordingly, we introduce the second ansatz

$$D(t,x)=\mathcal{D}(\theta,x),\quad B(t,x)=t\mathcal{B}(\theta,x),\quad P(t,x)=t\mathcal{P}(\theta,x),\quad \theta=t^2/2, \tag{11}$$

or, equivalently,

$$\mathcal{D}(\theta,x)=D(\sqrt{2\theta},x),\quad \mathcal{B}(\theta,x)=\frac{B(\sqrt{2\theta},x)}{\sqrt{2\theta}},\quad \mathcal{P}(\theta,x)=\frac{P(\sqrt{2\theta},x)}{\sqrt{2\theta}}.$$

Let us now state our main result.

Theorem 2. *After the two "dual" quadratic changes of time (10) and (11), the minimal surface equations, written in augmented form (6)–(8), respectively lead on one hand to the mean-curvature flow (3) and on the other hand to the arctangential heat equation (1).*

Let us emphasize the formal character of this result, where we are just dealing with the equations. We will not discuss in the present paper the analysis of how close the solutions of the minimal surface equations (2), written in augmented form (6)–(8), after quadratic change of time, are from the solutions of (1) and (3), respectively. Let us just indicate our belief that the "relative entropy" (or "modulated energy") method (as in Dafermos' book [2016], see also [Brenier and Duan 2018; Giesselmann et al. 2017; Serfaty 2017] for very recent occurrences), based on the strict convexity of (8) and the dissipation law (21) established below, is the most appropriate tool to treat this question.

Proof of Theorem 2. Let us transform the augmented system (6)–(8) in both regimes (10)–(11). In the first case, we get the nonautonomous system, where θ features explicitly,

$$\partial_\theta\mathcal{B}_i+\partial_i\left(\frac{\mathcal{B}_j\mathcal{P}^j-\mathcal{D}}{\mathcal{H}}\right)=0,\quad \mathcal{D}+2\theta\left(\partial_\theta\mathcal{D}+\partial_j\left(\frac{\mathcal{D}\mathcal{P}^j}{\mathcal{H}}\right)\right)=\partial_j\left(\frac{\mathcal{B}^j}{\mathcal{H}}\right),$$
$$\mathcal{P}^i+2\theta\left(\partial_\theta\mathcal{P}^i+\partial_j\left(\frac{\mathcal{P}^i\mathcal{P}^j}{\mathcal{H}}\right)\right)+\partial_j\left(\frac{\mathcal{B}^i\mathcal{B}^j}{\mathcal{H}}\right)=\partial^i\left(\frac{1+\mathcal{B}_j\mathcal{B}^j}{\mathcal{H}}\right),$$

where

$$\mathcal{H}=\sqrt{1+\mathcal{B}_j\mathcal{B}^j+2\theta(\mathcal{D}^2+\mathcal{P}_j\mathcal{P}^j)}.$$

Formally, this system admits, as $\theta \downarrow 0$, the following asymptotic system

$$\partial_\theta \mathcal{B}_i + \partial_i \left(\frac{\mathcal{B}_j \mathcal{P}^j - \mathcal{D}}{\sqrt{1+\mathcal{B}_k \mathcal{B}^k}} \right) = 0, \quad \mathcal{D} = \partial_j \left(\frac{\mathcal{B}^j}{\sqrt{1+\mathcal{B}_k \mathcal{B}^k}} \right), \tag{12}$$

$$\mathcal{P}^i = -\partial_j \left(\frac{\mathcal{B}^i \mathcal{B}^j}{\sqrt{1+\mathcal{B}_k \mathcal{B}^k}} \right) + \partial^i \left(\frac{1+\mathcal{B}_j \mathcal{B}^j}{\sqrt{1+\mathcal{B}_k \mathcal{B}^k}} \right). \tag{13}$$

In the second regime, when we rather assume $B(0, x) = P(0, x) = 0$ and use ansatz (11) instead of (10), we get from (6)–(8) again a nonautonomous system where θ features explicitly:

$$\partial_\theta \mathcal{D} + \partial_j \left(\frac{\mathcal{D}\mathcal{P}^j - \mathcal{B}^j}{\mathcal{H}} \right) = 0, \quad \mathcal{B}_i + 2\theta \left(\partial_\theta \mathcal{B}_i + \partial_i \left(\frac{\mathcal{B}_j \mathcal{P}^j}{\mathcal{H}} \right) \right) = \partial_i \left(\frac{\mathcal{D}}{\mathcal{H}} \right),$$

$$\mathcal{P}^i + 2\theta \left(\partial_\theta \mathcal{P}^i + \partial_j \left(\frac{\mathcal{P}^i \mathcal{P}^j + \mathcal{B}^i \mathcal{B}^j}{\mathcal{H}} \right) \right) = \partial^i \left(\frac{1}{\mathcal{H}} \right),$$

where

$$\mathcal{H} = \sqrt{1 + \mathcal{D}^2 + 2\theta(\mathcal{B}_j \mathcal{B}^j + \mathcal{P}_j \mathcal{P}^j)},$$

which admits as asymptotic system, as $\theta \downarrow 0$,

$$\partial_\theta \mathcal{D} + \partial_j \left(\frac{\mathcal{D}\mathcal{P}^j - \mathcal{B}^j}{\sqrt{1+\mathcal{D}^2}} \right) = 0, \quad \mathcal{B}_i = \partial_i \left(\frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}} \right), \tag{14}$$

$$\mathcal{P}_i = \partial_i \left(\frac{1}{\sqrt{1+\mathcal{D}^2}} \right). \tag{15}$$

Restoring notations (t, D, B, P), instead of $(\theta, \mathcal{D}, \mathcal{B}, \mathcal{P})$, we may write both asymptotic systems (12)–(13) and (14)–(15) respectively as

$$\partial_t B_i + \partial_i \left(\frac{B_j P^j - D}{\eta} \right) = 0, \quad \eta = \sqrt{1 + B_k B^k} \tag{16}$$

$$D = \partial_j \left(\frac{B^j}{\eta} \right), \quad P^i = -\partial_j \left(\frac{B^i B^j}{\eta} \right) + \partial^i \left(\frac{1 + B_j B^j}{\eta} \right) \tag{17}$$

and

$$\partial_t D + \partial_j \left(\frac{D P^j - B^j}{\eta} \right) = 0, \quad \eta = \sqrt{1 + D^2}, \tag{18}$$

$$B_i = \partial_i \left(\frac{D}{\eta} \right), \quad P_i = \partial_i \left(\frac{1}{\eta} \right). \tag{19}$$

Let us first derive the arctangential heat equation (1) from (18)–(19). We get

$$\begin{aligned}\partial_t D &= \partial_j\left(-\frac{DP^j}{\eta}+\frac{B^j}{\eta}\right) = \partial_j\left(-\frac{D}{\eta}\partial^j\left(\frac{1}{\eta}\right)+\frac{1}{\eta}\partial^j\left(\frac{D}{\eta}\right)\right)\\ &= \partial_j\left(\frac{\partial_j D}{\eta^2}\right) = \partial_j\left(\frac{\partial_j D}{1+D^2}\right) = \Delta(\arctan D).\end{aligned}$$

Let us now derive the mean curvature flow (3) from (16)–(17). Writing B as a gradient, i.e., $B_i = \partial_i\phi$, we may integrate (16) just as

$$\partial_t\phi = \frac{D-\partial_i\phi P^i}{\eta}. \tag{20}$$

We have

$$\begin{aligned}P^i &= -\partial_j\left(\frac{\partial^i\phi\,\partial^j\phi}{\eta}\right)+\partial^i\left(\frac{1+\partial_j\phi\,\partial^j\phi}{\eta}\right)\\ &= -\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi-\left(\frac{\partial^j\phi}{\eta}\right)\partial_j\partial^i\phi+\partial^i\left(\frac{1}{\eta}\right)+\partial^i\left(\frac{\partial_j\phi\,\partial^j\phi}{\eta}\right)\\ &= -\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi+\partial^i\left(\frac{1}{\eta}\right)+\partial_j\phi\,\partial^i\left(\frac{\partial^j\phi}{\eta}\right)\\ &= -\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi+\eta\,\partial^i\left(\frac{1}{2\eta^2}\right)+\eta\,\partial^i\left(\frac{\partial_j\partial^j\phi}{2\eta^2}\right)\\ &= -\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi+\eta\,\partial^i\left(\frac{1+\partial_j\partial^j\phi}{2\eta^2}\right)\\ &= -\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi\end{aligned}$$

(by definition of η). Thus, by (20)

$$\partial_t\phi = \frac{D}{\eta}+\frac{\partial_i\phi\,\partial^i\phi}{\eta}\,\partial_j\left(\frac{\partial^j\phi}{\eta}\right).$$

Since

$$D = \partial_j\left(\frac{\partial^j\phi}{\eta}\right),$$

we get

$$\partial_t\phi = \frac{1+\partial_i\phi\,\partial^i\phi}{\eta}\,\partial_j\left(\frac{\partial^j\phi}{\eta}\right) = \sqrt{1+\partial_i\phi\,\partial^i\phi}\,\partial_j\left(\frac{\partial^j\phi}{\sqrt{1+\partial_k\phi\,\partial^k\phi}}\right)$$

(by definition of η), which exactly is the mean curvature flow (3). This concludes the proof of Theorem 2.

Remark. It is worth noticing that, in the case of the second ansatz (11), leading to the arctangential equation (1) after quadratic change of time, the extra *conservation* law (9) leads to the *dissipation* law

$$\partial_t \eta + \Delta\left(\frac{1}{\eta}\right) = -\frac{B^k B_k + P^k P_k}{\eta}, \quad \eta = \sqrt{1+D^2}, \quad B_i = \partial_i\left(\frac{D}{\eta}\right), \quad P_i = \partial_i\left(\frac{1}{\eta}\right) \tag{21}$$

on top of conservation laws (18)–(19), which, consistently, can be as well obtained directly from (1).

3. A few properties of the arctangential heat equation

3.1. ***Interpretation à la Otto.*** Assuming that D is nonnegative (which is a consistent assumption due to the maximum principle for (1)), the arctangential heat equation also reads

$$\partial_t D = \partial_i\left(D\,\partial^i\left(\log\left(\frac{D}{\sqrt{1+D^2}}\right)\right)\right), \tag{22}$$

which can be written, in the framework of optimal transport theory [Ambrosio et al. 2008; Santambrogio 2015; Villani 2003],

$$\partial_t D = \partial_i\big(D\,\partial^i(\mathcal{F}'(D))\big), \tag{23}$$

à la Otto, as the gradient flow, with respect to the (so-called) "Wasserstein" or "MK2" metric [Otto 2001; Otto and Westdickenberg 2005], of the functional

$$\mathcal{D} \to \int \mathcal{F}(\mathcal{D}(x))\,dx$$

for a suitable function $\mathcal{F}$. Here $\mathcal{F}$ is a "renormalized" version of the classical Boltzmann entropy, namely

$$\mathcal{F}(D) = D\log\left(\frac{D}{\sqrt{1+D^2}}\right) - \arctan D \tag{24}$$

and should be extended by 0 for $D = 0$ and by $+\infty$ for $D < 0$ to define a globally convex function from $\mathbb{R}$ to $]-\infty, +\infty]$. Its Legendre–Fenchel transform can be explicitly (and easily) computed:

$$u \to \sup_D(uD - \mathcal{F}(D)) = (\mathbb{G}\exp)(u) = \arcsin(\exp(u)), \tag{25}$$

which should be extended by $+\infty$ for $u > 0$ and can be seen as a "generalized" exponential function. (Here the symbol $\mathbb{G}$ is used to note a generalization of a classical special function).

As a matter of fact, if we consistently parametrize the arctangential heat equation, in its formulation à la Otto (22), as

$$\partial_t D = \partial_i \left(D\, \partial^i \left(\log\left(\frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) \right) \right), \tag{26}$$

where $\lambda > 0$ should be understood as a large "cutoff" parameter, the corresponding Boltzmann entropy becomes

$$D \log\left(\frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) - \lambda \arctan(D\lambda^{-1}), \tag{27}$$

whose Legendre–Fenchel transform reads

$$(\mathbb{G}_\lambda \exp)(u) = \lambda \arcsin(\lambda^{-1} \exp(u)), \tag{28}$$

(extended by $+\infty$ for $u > \log \lambda$); see Figure 1. The later function is clearly an approximation of the regular exponential function as λ goes to infinity, with the interesting feature that, at $u = \log \lambda$, it reaches a finite value, namely $\lambda\pi/2$, and suddenly jumps to $+\infty$, while its u-derivative blows up. In some sense, $y = (\mathbb{G}_\lambda \exp)(u)$, which solves the super-nonlinear ODE

$$\frac{dy}{du} = \lambda \tan\left(\frac{y}{\lambda} \right),$$

while its derivative $z = \frac{dy}{du}$, which solves

$$\frac{dz}{du} = \left(1 + \frac{z^2}{\lambda^2} \right) z,$$

is a "catastrophic" version of the exponential function, probably suitable for some applications in geophysics, biology, social sciences and many other fields. Also notice that the inverse of this generalized exponential function provides a generalization of the logarithm, namely,

$$(\mathbb{G}_\lambda \log)(v) = \log(\lambda \sin(v\lambda^{-1})).$$

This function monotonically covers $]-\infty, \log \lambda]$ as $v \in]0, \lambda\pi/2[$ and can be symmetrically and periodically extended to $v \in \mathbb{R}$ as

$$(\mathbb{G}_\lambda \log)(v) = \tfrac{1}{2} \log(\lambda^2 \sin^2(v\lambda^{-1}));$$

see Figure 2. This features in several fields of Mathematics, including the recent theory of "unbalanced optimal transportation" [Chizat et al. 2018; Liero et al. 2018].

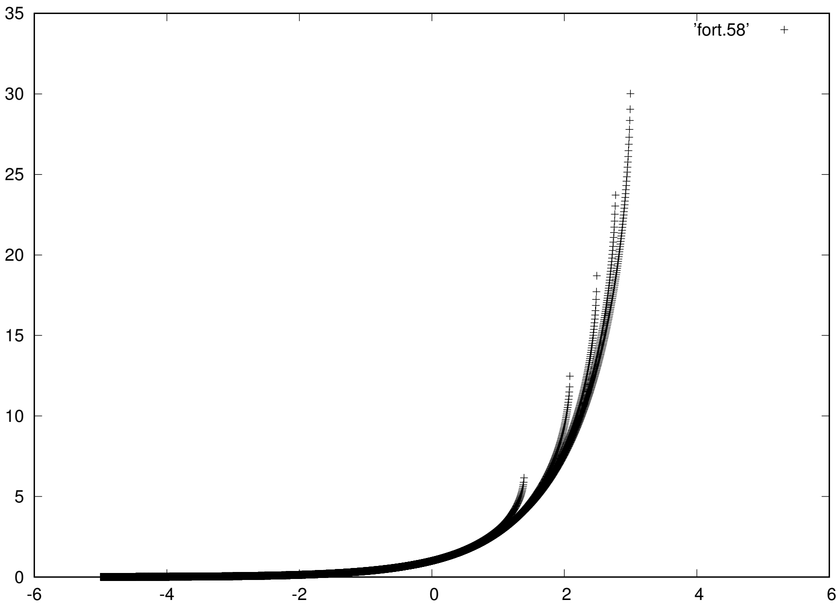

Figure 1. The "catastrophic" exponential $u \to \lambda \arcsin(\lambda^{-1} \exp u)$ (extended by $+\infty$ for $u > \log \lambda$), for different values of λ.

3.2. ***A limit case: the Chaplygin heat equation.*** The arctangential heat equation, in its parametrized form (26), namely

$$\partial_t D = \partial_i \left(D\, \partial^i \left(\log \left(\frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) \right) \right),$$

admits an interesting formal limit as $\lambda \downarrow 0$, Indeed, we have

$$\partial^i \left(\log \left(\frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) \right) = \partial^i \left(\log \left(\frac{\lambda}{\sqrt{1 + \lambda^2 D^{-2}}} \right) \right) = \partial^i \left(\log \left(\frac{1}{\sqrt{1 + \lambda^2 D^{-2}}} \right) \right)$$
$$\sim -\partial^i \left(\frac{\lambda^2 D^{-2}}{2} \right) = -\lambda^2 D^{-1} \partial^i (D^{-1}), \quad \lambda \downarrow 0,$$

so that, as $\lambda \downarrow 0$, after rescaling $t \to \lambda^2 t$, we get from (26) the "Chaplygin heat equation",

$$\partial_t D = -\Delta(D^{-1}). \tag{29}$$

(Notice that this equation can also be directly obtained from the Euler equations of "isentropic fluids",

$$\partial_t D + \partial_k Q^k = 0, \quad \partial_t Q^i + \partial_k \left(\frac{Q^k\, Q^i}{D} \right) + \partial^i (p(D)) = 0,$$

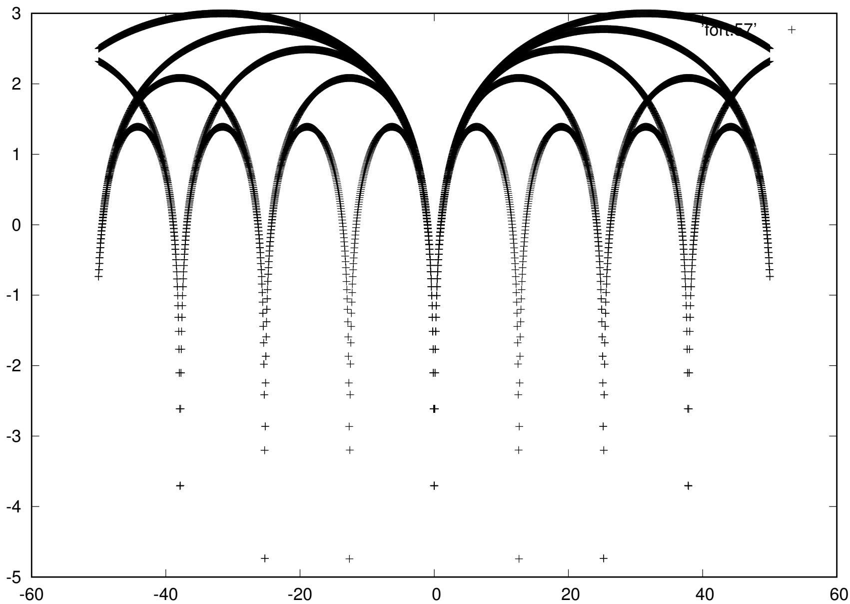

Figure 2. The inverse of the "catastrophic" exponential $v \to \frac{1}{2}\log(\lambda^2 \sin^2(v\lambda^{-1}))$, (after symmetrization and periodization) for different values of λ.

with "Chaplygin pressure" $p(D) = -D^{-1}$ (as in [Serre 2009]), after the same quadratic change of time

$$D(t,x) = \mathcal{D}(\theta, x), \quad Q(t,x) = t\mathcal{Q}(\theta, x), \quad \theta = t^2/2,$$

we used above.)

3.3. ***Relationship with nonlinear electromagnetism in two space dimensions.*** Let us now show that, in the case of two space dimensions, $d = 2$, both equations (3) and (1) can be (formally) derived from one of the most famous (and very geometric) models of nonlinear electromagnetism, namely the Born–Infeld equations, again through a suitable quadratic change of time. We first notice that these equations just describe particular solutions, depending only on two space variables, of the *same* three-dimensional vectorial diffusion equation:

Proposition 3. *In two space dimensions, both the mean curvature flow (3) and the arctangential heat equation (1) correspond to special solutions of the three-dimensional diffusion equation*

$$\partial_t D = \nabla \times \left(B\sqrt{1+D^2} - \frac{(D\cdot B)D}{\sqrt{1+D^2}} \right), \quad B = -\nabla \times \left(\frac{D}{\sqrt{1+D^2}} \right). \tag{30}$$

Proof. Straightforward calculations show that (3) just corresponds to particular solutions D of form

$$D(t,x) = (-\partial_2\phi(t,x^1,x^2), \partial_1\phi(t,x^1,x^2), 0),$$

while (1) rather corresponds to solutions of "dual" form

$$D(t,x) = (0,0,D(t,x^1,x^2)),$$

which, in both cases, implies $B \cdot D = 0$ and leads to, respectively, (3) and (1). □

The augmented Born–Infeld system. Next, we derive (30) from a suitable quadratic change of time, as a formal asymptotic equation for the nonlinear Maxwell equations

$$\partial_t B + \nabla \times \left(\frac{\partial H}{\partial D}(D,B)\right) = 0, \quad \partial_t D - \nabla \times \left(\frac{\partial H}{\partial B}(D,B)\right) = 0, \tag{31}$$

where the Hamiltonian function H is

$$H(D,B) = \sqrt{(1+B^2)(1+D^2) - (D\cdot B)^2}$$

(while $H(D,B) = (B^2+D^2)/2$ would correspond to the usual, linear, Maxwell equations). This nonlinear correction to the Maxwell equations was suggested by Born and Infeld [1934]. Let us write (31) more explicitly. Introducing

$$P = D\times B, \quad h = \sqrt{(1+D^2)(1+B^2) - (D\cdot B)^2} = \sqrt{1+D^2+B^2+P^2},$$

we get

$$\partial_t B + \nabla\cdot\left(\frac{B\otimes P - P\otimes B}{h}\right) = -\nabla\times\left(\frac{D}{h}\right), \tag{32}$$

$$\partial_t D + \nabla\cdot\left(\frac{D\otimes P - P\otimes D}{h}\right) = \nabla\times\left(\frac{B}{h}\right). \tag{33}$$

As in Section 1, this system of conservation laws admits an extra conservation law for h. However h, as a function of (D,B), is convex only about $(0,0)$ and not globally. Following [Brenier 2004; Serre 2004], again as in Section 1, we get a new extra conservation law by considering P as an independent variable and write h as a function of (D,B,P):

$$h = h(D,B,P) = \sqrt{1+D^2+B^2+P^2}. \tag{34}$$

We obtain

$$\partial_t P + \nabla\cdot\left(\frac{P\otimes P - B\otimes B - D\otimes D}{h}\right) = \nabla\left(\frac{1}{h}\right). \tag{35}$$

In [Brenier 2004; Serre 2004], it is proven that the augmented system (32)–(35) enjoys an extra conservation law for h written as a function of (D,B,P).

The diffusive limit of the augmented Born–Infeld system.

Proposition 4. *The diffusion equation (30) can be obtained from the augmented Born–Infeld system (32)–(35) after the quadratic change of time*

$$D(t,x)=\mathcal{D}(\theta,x),\quad B(t,x)=t\mathcal{B}(\theta,x),\quad P(t,x)=t\mathcal{P}(\theta,x),\quad \theta=t^2/2, \tag{36}$$

Proof. Let us apply ansatz (36) to system (32)–(35). We get the nonautonomous system, where θ features explicitly,

$$\mathcal{B}+2\theta\left(\partial_\theta\mathcal{B}+\nabla\cdot\left(\frac{\mathcal{B}\otimes\mathcal{P}-\mathcal{P}\otimes\mathcal{B}}{\mathcal{H}}\right)\right)=-\nabla\times\left(\frac{\mathcal{D}}{\mathcal{H}}\right),$$
$$\partial_\theta\mathcal{D}+\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{P}-\mathcal{P}\otimes\mathcal{D}}{\mathcal{H}}\right)=\nabla\times\left(\frac{\mathcal{B}}{\mathcal{H}}\right),$$
$$\mathcal{P}+2\theta\left(\partial_\theta\mathcal{P}+\nabla\cdot\left(\frac{\mathcal{P}\otimes\mathcal{P}-\mathcal{B}\otimes\mathcal{B}}{\mathcal{H}}\right)\right)=\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{D}}{\mathcal{H}}\right)+\nabla\left(\frac{1}{\mathcal{H}}\right),$$

where

$$\mathcal{H}=\sqrt{1+\mathcal{D}^2+2\theta(\mathcal{B}^2+\mathcal{P}^2)}.$$

As $\theta\downarrow 0$, we get the asymptotic system

$$\partial_\theta\mathcal{D}+\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{P}-\mathcal{P}\otimes\mathcal{D}}{\mathcal{H}}\right)=\nabla\times\left(\frac{\mathcal{B}}{\mathcal{H}}\right),$$
$$\mathcal{B}=-\nabla\times\left(\frac{\mathcal{D}}{\mathcal{H}}\right),\quad \mathcal{P}=\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{D}}{\mathcal{H}}\right)+\nabla\left(\frac{1}{\mathcal{H}}\right),$$

where, now,

$$\mathcal{H}=\sqrt{1+\mathcal{D}^2}.$$

The equality $\mathcal{P}=\mathcal{D}\times\mathcal{B}$ follows directly from these equations. Thus, since we are in three space dimensions,

$$\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{P}-\mathcal{P}\otimes\mathcal{D}}{\mathcal{H}}\right)=\nabla\times\left(\frac{\mathcal{D}\times\mathcal{P}}{\mathcal{H}}\right)=\nabla\times\left(\frac{\mathcal{D}\times(\mathcal{D}\times\mathcal{B})}{\mathcal{H}}\right)=\nabla\times\left(\frac{(\mathcal{D}\cdot\mathcal{B})\mathcal{D}-\mathcal{D}^2\mathcal{B}}{\mathcal{H}}\right).$$
$$=\nabla\times\left(\frac{(\mathcal{D}\cdot\mathcal{B})\mathcal{D}-(1+\mathcal{D}^2)\mathcal{B})}{\mathcal{H}}\right)+\nabla\times\left(\frac{\mathcal{B}}{\mathcal{H}}\right),$$
$$=\nabla\times\left(\frac{(\mathcal{D}\cdot\mathcal{B})\mathcal{D}}{\mathcal{H}}-\mathcal{H}\mathcal{B}\right)+\nabla\times\left(\frac{\mathcal{B}}{\mathcal{H}}\right).$$

Finally, we have found

$$\partial_\theta\mathcal{D}=\nabla\times\left(\mathcal{B}\sqrt{1+\mathcal{D}^2}-\frac{(\mathcal{D}\cdot\mathcal{B})\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}\right),\quad \mathcal{B}=-\nabla\times\left(\frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}\right)$$

which is nothing but the expected Equation (30), after restoring notations (t, B, D) instead of $(\theta, \mathcal{B}, \mathcal{D})$. □

4. The arctangential heat equation in nonconservative form: a tool for image processing?

In nonconservative form, the arctangential heat equation (1) reads (4), namely

$$\partial_t \psi = \left(\frac{\cos(\pi\psi)}{\pi}\right)^2 \Delta\psi.$$

Interestingly enough, in this nonconservative formulation, ψ can take values in the entire real line and not only in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In sharp contrast with the usual linear heat equation, (4) seems to admit (in a suitable sense) a lot of nontrivial equilibrium solutions, at least in the one dimension case $d = 1$. Such solutions ψ should be continuous piecewise linear functions, with possible change of slope (or plateaus) each time ψ touches the discrete set $\left\{k + \frac{1}{2},\ k \in \mathbb{Z}\right\}$ as $\cos(\pi\psi)$ vanishes. Let us now perform a few numerical experiments based on the very elementary explicit difference scheme (written in two space dimensions with traditional notation of numerical analysis):

$$\psi_{i,j}^{n+1} - \psi_{i,j}^{n} = \frac{4\tau}{h^2}\cos(\pi\psi_{i,j}^{n})^2\left(\frac{\psi_{i+1,j}^{n} + \psi_{i-1,j}^{n} + \psi_{i,j+1}^{n} + \psi_{i,j-1}^{n}}{4} - \psi_{i,j}^{n}\right), \tag{37}$$

where τ and h respectively denote the time and space steps. This scheme is stable as long as $4\tau h^{-2} \le 1$ and, in all our numerical experiments, we will choose $4\tau = h^2$.

In the first experiment, we input as initial condition $\psi(0, \cdot)$ a (simulated) Brownian curve on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (made periodic by subtracting a suitable affine function). We draw the initial curve (Figure 3) and the final curve (Figure 4) obtained with 256 grid points after 4096 time steps.

The second experiment is of different nature. We use a 256×256 grid for the periodic square $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and consider the function

$$\begin{aligned}\psi&(0, x, y)\\ &= 4\cos(2\pi(x - 0.25))\cos(2\pi(y - 0.2)) + 3\cos(2\pi(y + x))\cos(2\pi(x - 0.8)).\end{aligned}$$

Then we add to $\psi(0, x, y)$, at each grid point, a random number ξ, uniformly independently distributed in $[-0.5, +0.5]$ and run (37). In three successive plots, Figures 5–7, we draw all grid points (i, j) where the resulting function is at a distance less than 0.025 from the set $\left\{k + \frac{1}{2},\ k \in \mathbb{Z}\right\}$ in $\mathbb{R}$, respectively for $n = 0$, first without noise (Figure 5), then with added noise (Figure 6), and, finally, for $n = 8192$ (Figure 7), i.e., after 8192 time steps. We see that the arctangential heat equation, in discretized nonconservative form (37), enjoys some ability at processing black and white images, $\psi_{i,j}^n$ being the (suitably normalized) level of gray at step n and grid point (i, j), by unveiling and enhancing the level sets $\left\{(i, j),\ \psi_{i,j}^n \in \left\{k + \frac{1}{2},\ k \in \mathbb{Z}\right\}\right\}$ as n grows.

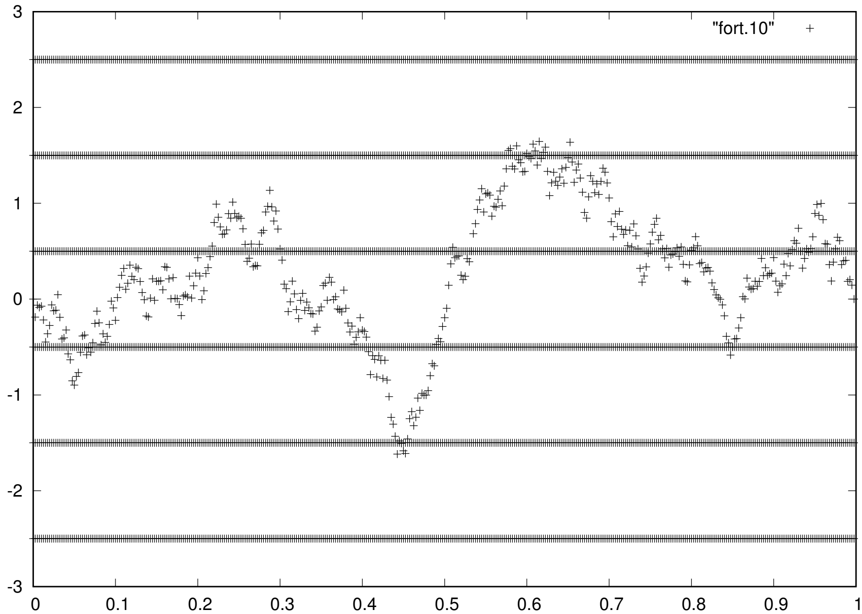

Figure 3. The one-dimensional arctangential heat equation: brownian initial condition, $x \in \mathbb{R}/\mathbb{Z}$.

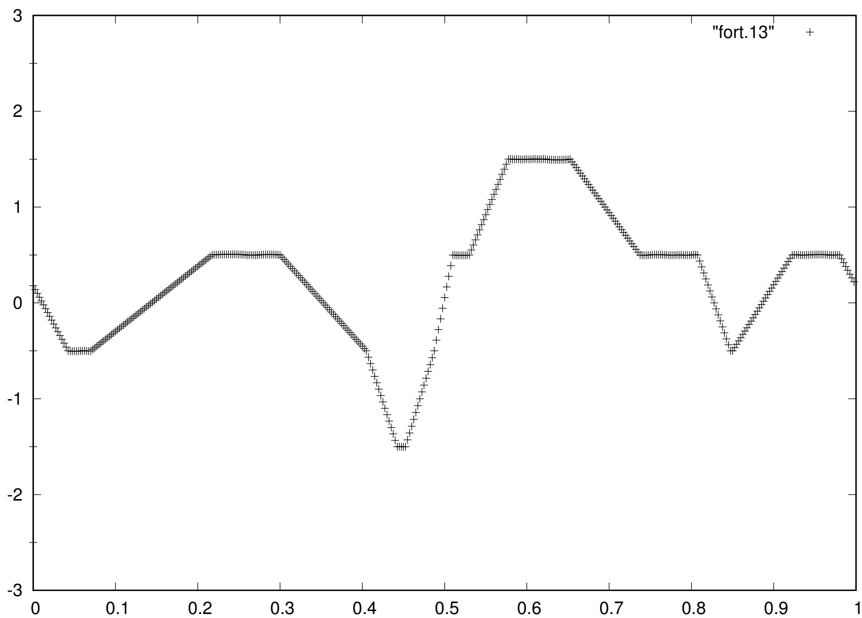

Figure 4. Numerical solution for 256 grid points, 4096 time steps.

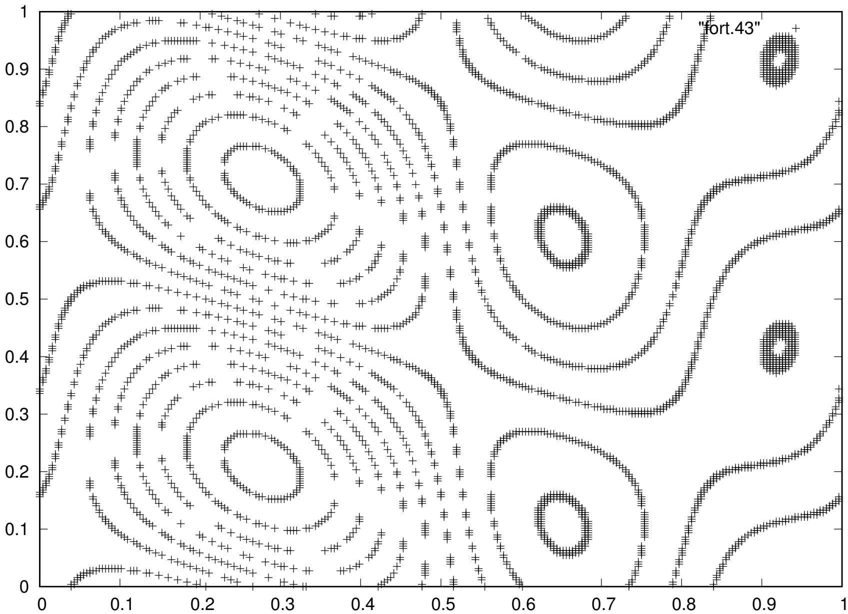

Figure 5. The two-dimensional arctangential heat equation: level sets of the given function with no noise.

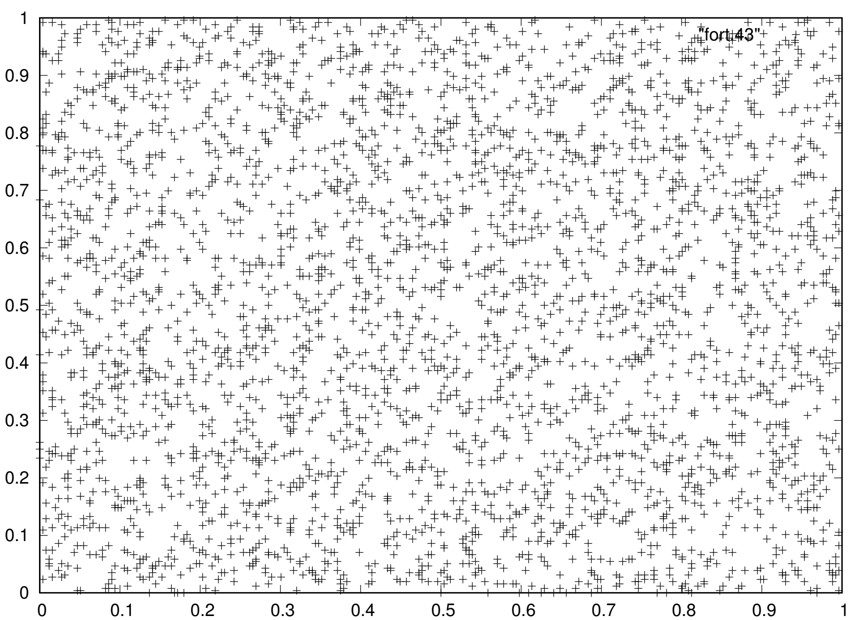

Figure 6. The two-dimensional arctangential heat equation: the initial condition with added noise.

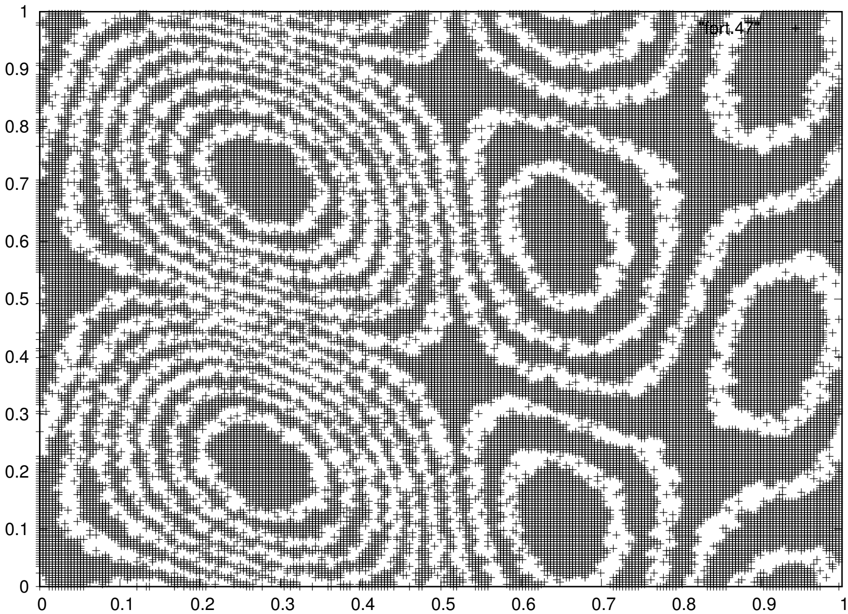

Figure 7. Recovery of the level sets by solving the two-dimensional arctangential heat equation.

Finally, in the third experiment, our initial condition is obtained by adding the same noise ξ as before to the step function with values $\frac{1}{2}$ if

$$4\cos(2\pi(x-0.25))\cos(2\pi(y-0.2))+3\cos(2\pi(y+x))\cos(2\pi(x-0.8))\geq 0.5,$$

and 0 otherwise. We draw the same plots as in the previous experiment, this time for $n=0$ (with and without noise, Figures 8 and 9) and $n=1024$ (Figure 10).

Appendix: Proof of Theorem 1

First step: Hamiltonian form of the minimal surface equations. Equation (2) is easily obtained by finding critical points ϕ of the Minkowski area of the graph $(t,x)\to(t,x,\phi(t,x))$, namely

$$-\iint\sqrt{1-\partial_t\phi^2+\partial^k\phi\,\partial_k\phi}\,dt\,dx, \tag{38}$$

under space-time compactly supported perturbations. For the sequel, it is crucial to use the Hamiltonian form of Equation (2). For that purpose, we introduce the fields

$$E(t,x)=\partial_t\phi(t,x),\qquad B_i(t,x)=\partial_i\phi(t,x), \tag{39}$$

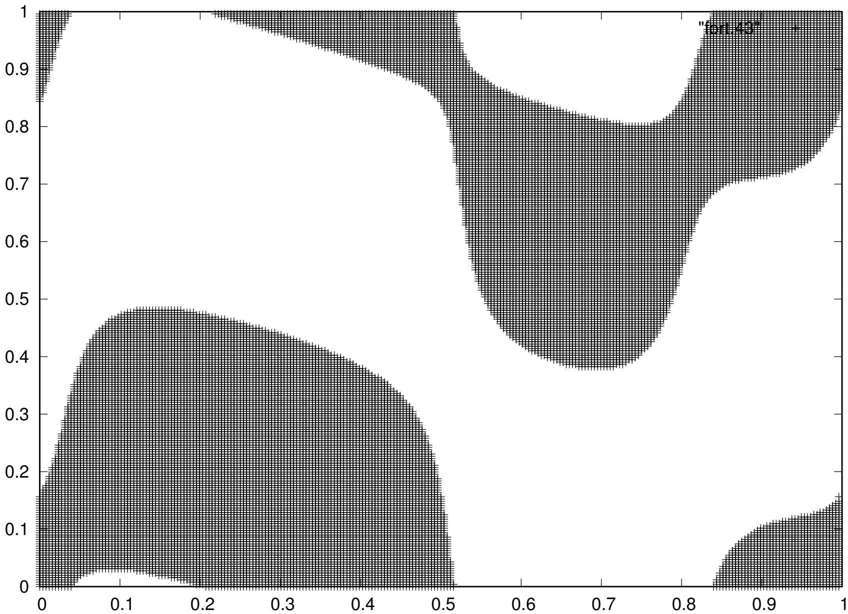

Figure 8. The two-dimensional arctangential heat equation: other choice of data (with binary values).

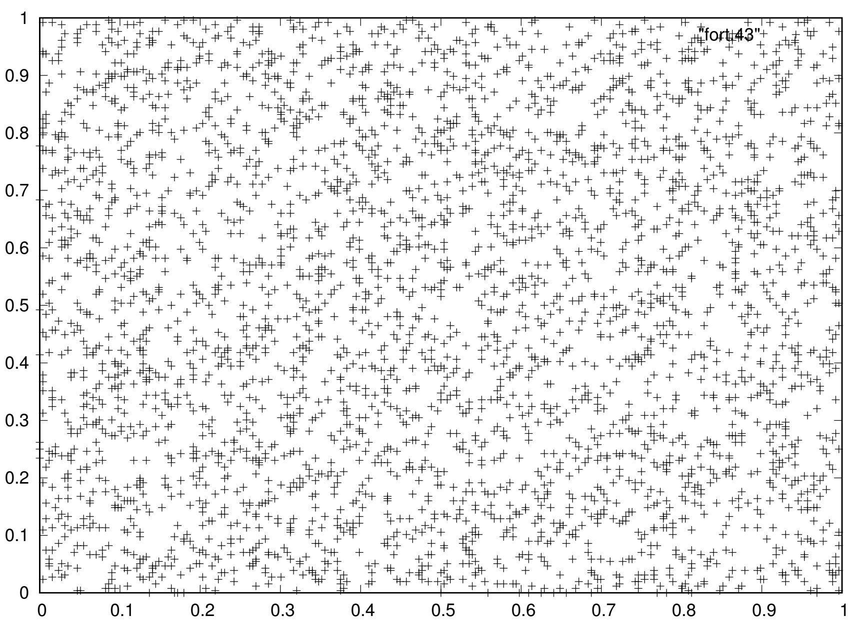

Figure 9. The two-dimensional arctangential heat equation: the initial condition with added noise.

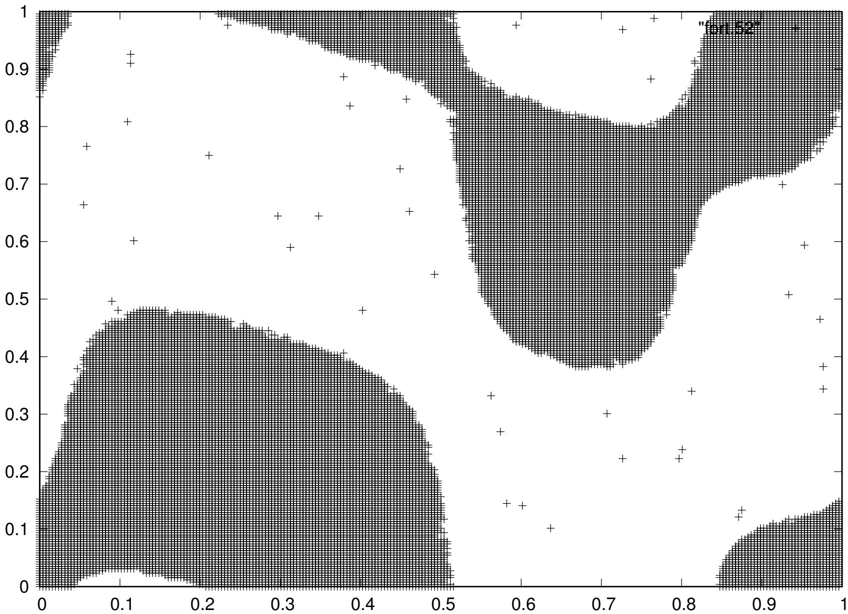

Figure 10. Recovery of the level sets by solving the two-dimensional arctangential heat equation.

which are linked by the differential compatibility condition

$$\partial_t B_i = \partial_i E. \tag{40}$$

Introducing the Lagrangian function

$$L(E, B) = -\sqrt{1 - E^2 + B_k B^k}, \tag{41}$$

we look at critical points (E, B) of

$$\iint L(E(t, x), B(t, x))\, dt\, dx$$

under space-time compactly supported perturbations, subject to constraint (40). In other words, we look for saddle-points (E, B, ψ) of

$$\iint \big(L(E(t, x), B(t, x)) + \partial_t \psi^i B_i(t, x) - \partial_i \psi^i E(t, x)\big)\, dt\, dx,$$

where ψ is a Lagrange multiplier for constraint (40). Independently of the specific definition of L, we may introduce the Hamiltonian H as the partial Legendre–Fenchel transform of the Lagrangian $L(E, B)$ with respect to E,

$$H(D, B) = \sup_{E \in \mathbb{R}} DE - L(E, B) \tag{42}$$

and the corresponding "dual" field

$$D(t,x) = \left(\frac{\partial L}{\partial E}\right)(E(t,x), B(t,x)). \tag{43}$$

Then, we get, by standard differential calculus, the Hamiltonian formulation

$$\partial_t B_i = \partial_i \left(\frac{\partial H}{\partial D}(D,B)\right), \quad \partial_t D = \partial_i \left(\frac{\partial H}{\partial B_i}(D,B)\right), \tag{44}$$

and, as a consequence, an extra conservation law involving H:

$$\partial_t(H(D,B)) + \partial_i(P^i(D,B)) = 0, \quad P^i(D,B) = -\left(\frac{\partial H}{\partial D}\frac{\partial H}{\partial B_i}\right)(D,B). \tag{45}$$

In the case of the minimal surface equations L is given by (41) and we get, explicitly,

$$H(D,B) = \sqrt{(1+B_k\,B^k)(1+D^2)} \tag{46}$$

and, after elementary calculations, deduce:

Proposition 5. *The minimal surface equations* (2) *can be written in Hamiltonian form*

$$\partial_t B_i = \partial_i \left(\sqrt{\frac{1+B_kB^k}{1+D^2}}\,D\right), \quad \partial_t D = \partial_i \left(\sqrt{\frac{1+D^2}{1+B_kB^k}}\,B^i\right), \tag{47}$$

with the extra conservation law

$$\partial_t H + \partial_i P^i = 0, \quad H = \sqrt{(1+B_k\,B^k)(1+D^2)}, \quad P^i = -DB^i. \tag{48}$$

In addition, (D,B) *are related to the original field* ϕ *involved in* (2) *by*

$$B_i = \partial_i\phi, \quad D = \frac{\partial_t\phi}{\sqrt{1-\partial_t\phi^2+\partial^k\phi\,\partial_k\phi}}. \tag{49}$$

Second step: construction of an augmented system with convex entropy. Unfortunately, H, as defined by (46), is not a convex function of (D,B) and, therefore, (47) does not belong to the class of systems of "conservation laws with a convex entropy" which enjoys many interesting properties (as discussed in Dafermos' book [2016]). However, there is also an extra conservation law for $P=-DB$, namely (7). This allows (D,B,P) to be solution of the *augmented* system (6)–(7) of conservation laws which enjoys the extra conservation law (9) for the strictly convex "entropy" $h(D,B,P) = \sqrt{1+D^2+B_kB^k+P_kP^k}$, which is nothing but $H(D,B)$, written as a function of (D,B,P). Let us now provide the detailed calculations.

The first evolution equations (6) are straightforward (just writing (47) with $P=-DB$). The two last ones are much more involved. Let us first prove (7).

Since $P_i = -DB_i$, we get

$$\partial_t P_i = -D\partial_t B_i - B_i \partial_t D = T = T_4 + T_3 + T_1 + T_2,$$

$$T_4 = D\partial_i\left(\frac{B_j P^j}{h}\right), \quad T_3 = -D\partial_i\left(\frac{D}{h}\right), \quad T_1 = B_i\partial_j\left(\frac{DP^j}{h}\right), \quad T_2 = -B_i\partial_j\left(\frac{B^j}{h}\right),$$

using Theorem 1. We have

$$\begin{aligned} T_4 &= T_4a + T_4b, & T_4a &= DB_j\partial_i\left(\frac{P^j}{h}\right), & T_4b &= \frac{DP^j}{h}\partial_i B_j, \\ T_3 &= T_3a + T_3b, & T_3a &= -\partial_i\left(\frac{D^2}{h}\right), & T_3b &= \frac{D}{h}\partial_i D, \\ T_1 &= T_1a + T_1b, & T_1a &= \partial_j\left(\frac{B_i DP^j}{h}\right), & T_1b &= -\partial_j B_i \frac{DP^j}{h}, \\ T_2 &= T_2a + T_2b, & T_2a &= -\partial_j\left(\frac{B_i B^j}{h}\right), & T_2b &= \partial_j B_i \frac{B^j}{h}. \end{aligned}$$

Since $P_j = -DB_j$, we have

$$T_1a = -\partial_j\left(\frac{P_i P^j}{h}\right),$$

$$T_4a = -P_j\partial_i\left(\frac{P^j}{h}\right) = T_4aa + T_4ab, \quad T_4aa = -\partial_i\left(\frac{P_j P^j}{h}\right), \quad T_4ab = \frac{P^j}{h}\partial_i P_j.$$

Since B is a gradient, we have $\partial_i B_j = \partial_j B_i$ and, therefore,

$$T_4b = -T_1b, \quad T_2b = \partial_i B_j \frac{B^j}{h},$$

so that

$$T_3b + T_2b + T_4ab = \frac{1}{2h}\partial_i(1 + D^2 + B_j B^j + P_j P^j) = \partial_i h = \partial_i\left(\frac{h^2}{h}\right)$$

(by definition (8) of h). Collecting all terms, we find

$$\begin{aligned} \partial_t P_i = T &= T_4aa + T_4ab + T_4b + T_3a + T_3b + T_1a + T_1b + T_2a + T_2b \\ &= T_4aa + T_3a + T_1a + T_2a + \partial_i h \\ &= -\partial_i\left(\frac{P_j P^j}{h}\right) - \partial_i\left(\frac{D^2}{h}\right) - \partial_j\left(\frac{P_i P^j}{h}\right) - \partial_j\left(\frac{B_i B^j}{h}\right) + \partial_i\left(\frac{h^2}{h}\right) \\ &= \partial_i\left(\frac{1 + B_j B^j}{h}\right) - \partial_j\left(\frac{P_i P^j}{h}\right) - \partial_j\left(\frac{B_i B^j}{h}\right) \end{aligned}$$

(by definition (8) of h) and we have obtained (7). Let us now prove (9). Notice that, from now on, we no longer can use $P = -DB$. This equation should only follow from the augmented system (6)–(8) and the property that B is a gradient. Using definition (8) of h, we get

$$\begin{aligned}\partial_t h &= \frac{D\partial_t D + B^i\partial_t B_i + P^i\partial_t P_i}{h}\\ &= \frac{D}{h}\partial_j\left(\frac{-DP^j + B^j}{h}\right) + \frac{B^i}{h}\partial_i\left(\frac{-B_jP^j + D}{h}\right)\\ &\qquad + \frac{P^i}{h}\left(\partial_i\left(\frac{1 + B_jB^j}{h}\right) - \partial_j\left(\frac{P_iP^j}{h}\right) - \partial_j\left(\frac{B_iB^j}{h}\right)\right)\\ &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7,\end{aligned}$$

where

$$T_1 = \frac{D}{h}\partial_j\left(\frac{-DP^j}{h}\right), \qquad T_2 = \frac{D}{h}\partial_j\left(\frac{B^j}{h}\right)$$

$$T_3 = \frac{B^i}{h}\partial_i\left(\frac{-B_jP^j}{h}\right), \qquad T_4 = \frac{B^i}{h}\partial_i\left(\frac{D}{h}\right)$$

$$T_5 = \frac{P^i}{h}\partial_i\left(\frac{1 + B_jB^j}{h}\right)$$

$$T_6 = -\frac{P^i}{h}\partial_j\left(\frac{P_iP^j}{h}\right), \qquad T_7 = -\frac{P^i}{h}\partial_j\left(\frac{B_iB^j}{h}\right) = -\frac{P^j}{h}\partial_i\left(\frac{B_jB^i}{h}\right)$$

We see that

$$T_2 + T_4 = \partial_i\left(\frac{DB^i}{h^2}\right),$$

and

$$\begin{aligned}T_1+T_6 &= -P^j\left(\frac{D}{h}\partial_j\left(\frac{D}{h}\right)+\frac{P^i}{h}\partial_j\left(\frac{P_i}{h}\right)\right)-\partial_jP^j\left(\frac{D^2+P^2}{h^2}\right)\\ &= P^j\left(\frac{1}{h}\partial_j\left(\frac{1}{h}\right)+\frac{B^i}{h}\partial_j\left(\frac{B_i}{h}\right)\right)+\partial_jP^j\left(\frac{1+B_iB^i}{h^2}-1\right) \qquad \text{(definition of } h\text{)}\\ &= \frac{P^j}{h}\partial_j\left(\frac{1}{h}\right)+\frac{P^jB^i}{h^2}\partial_jB_i+\frac{P^jB_iB^i}{h}\partial_j\left(\frac{1}{h}\right)+\partial_jP^j\left(\frac{1+B_iB^i}{h^2}-1\right)\\ &= \frac{P^jB^i}{h^2}\partial_jB_i+\frac{P^j(1+B_iB^i)}{h}\partial_j\left(\frac{1}{h}\right)+\partial_jP^j\left(\frac{1+B_iB^i}{h^2}\right)-\partial_jP^j\\ &= \frac{P^jB^i}{h^2}\partial_jB_i+\partial_j\left(\frac{P^j(1+B_iB^i)}{h^2}\right)-T_5-\partial_jP^j.\end{aligned}$$

We also have

$$T_3+T_7=-\partial_i\left(\frac{P^jB_jB^i}{h^2}\right)-\frac{B^iP^j}{h^2}\partial_iB_j,$$

so that (since B is a gradient)

$$T_1+T_6+T_3+T_7+T_5=\partial_j\left(\frac{P^j(1+B_iB^i)}{h^2}\right)-\partial_i\left(\frac{P^jB_jB^i}{h^2}\right)-\partial_jP^j$$

and we have finally obtained

$$\begin{aligned}\partial_th&=T_1+T_2+T_3+T_4+T_5+T_6+T_7\\&=\partial_j\left(\frac{P^j(1+B_iB^i)}{h^2}\right)-\partial_i\left(\frac{P^jB_jB^i}{h^2}\right)+\partial_i\left(\frac{DB^i}{h^2}\right)-\partial_jP^j,\end{aligned}$$

in other words,

$$\partial_th+\partial_j\left(P^j-\frac{(DB^j+P^j)+B_k(B^kP^j-P^kB^j)}{h^2}\right)=0,$$

which is the desired conservation law (9) and achieves the proof of Theorem 1.

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