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We study a variation of the game of best choice (also known as the secretary problem or game of googol) under an additional assumption that the ranks of interview candidates are restricted using permutation pattern-avoidance. We describe the optimal positional strategies and develop formulas for the probability of winning.

1. Introduction

The game of best choice, also known as the “secretary problem,” appeared in Martin Gardner’s 1960 *Scientific American* column (reprinted in [Gardner 1995]), although it has a history which predates this; see, e.g., [Kadison 1994]. Gilbert and Mosteller [1966] gave a nice survey of the problem and solved some variations. The basic idea is to try to hire the best candidate out of N applicants for a job, each candidate having a specific ranking 1 (worst) through N (best). When interviewing the candidates, the decision must be made to hire them or not, on the spot, and candidates cannot be recalled later. The order of the interviews is (uniformly) random and so the interviewer does not know when the top candidate will come in.

As an example, suppose the interviews have rank order 574239618. The interviewer will be able to rank each initial segment of candidates relative to each other, but will not know their rank overall out of N . So the interviewer will see

1, 12, 231, 3421, 45312, 453126, ...

and must decide when to stop and hire. We count the game as a win if the best candidate out of N is hired and as a loss otherwise, with all losses having equal value. The optimal strategy, for N sufficiently large, turns out to be to reject the first N/e of the candidates (about 37%) and then hire the next candidate who is better than all earlier candidates.

Now, suppose that a consulting firm (with some oracular powers) agrees to filter candidates for the interviewer. They offer two strategies. In the first strategy, they

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will guarantee that each time a candidate B ranks lower than some candidate A already interviewed (“disappointing”), no future candidates will rank lower than B . In the second strategy, they guarantee that each time a candidate B ranks higher than some candidate A already interviewed (“raising the bar”), no future candidates will rank lower than A . All other aspects of the game remain the same.

Is there any difference between these? Are they better or worse than the classical case?

2. Refinement

Interview rank orders are *permutations* of some fixed size N which we write using the notation $p_1 p_2 \cdots p_N$, where the p_i are the values $1, 2, \dots, N$ arranged in some order. In this work, we restrict the interview rank orders using pattern-avoidance.

Definition 2.1. We say that the permutation $p = p_1 p_2 \cdots p_N$ contains the pattern $q = q_1 q_2 q_3$ if there exist $i < j < k$ such that p_i, p_j, p_k are in the same relative order as q_1, q_2, q_3 .

So, the “disappointment-free” consulting strategy is equivalent to requiring the interview rank orders to be 321-avoiding. Similarly, the “bar-raising” situation is the same as 231-avoiding. See the textbook [Bóna 2012] for a gentle introduction to pattern-avoidance. Putting aside the story about the consultants, we believe that pattern-avoidance is a natural mechanism for modeling the effect of domain learning by the player during the game. More precisely, as the interviewer ranks the current candidates at each step, they acquire information that allows them to hone the pool to include more relevant candidates at future time steps. We represent this honing process using pattern-avoidance.

The *left-to-right maxima* in a permutation p consist of elements p_j that are larger in value than every element p_i to the left (i.e., for $i < j$). In the game of best choice, it is never optimal to select a candidate that is not a left-to-right maximum. A *positional* strategy for the game of best choice is one in which the interviewer transitions from rejection to hiring based on the position of the interview. More precisely, the interviewer may play the k -positional strategy on a permutation p by rejecting candidates p_1, p_2, \dots, p_k and then accepting the next left-to-right maximum thereafter. If k is set too high, it is likely the player will miss the best candidate. If k is set too low, they will probably not have set their standards high enough to capture the best candidate. We say that a particular interview rank order is k -winnable if transitioning from rejection to hiring after the k -th interview captures the best candidate. For example, 574239618 is k -winnable for $k = 2, 3, 4$, and 5. It is straightforward to verify that a permutation p is k -winnable precisely when k lies between the last two left-to-right maxima in p .

In this paper, we restrict to using these positional strategies applied to a permutation chosen uniformly at random among those avoiding 321 (or, alternatively, 231) in order to facilitate comparison with the classical case. For each model, we seek to determine the optimal transition position k and probability of winning for finite N and asymptotically as $N \rightarrow \infty$.

We now mention some ties to recent work. Several authors have investigated the distribution of various permutation statistics for a random model in which a pattern-avoiding permutation is chosen uniformly at random. For example, [Miner and Pak 2014] finds the positions of smallest and largest elements as well as the number of fixed points in a random permutation avoiding a single pattern of size 3; [Madras and Pehlivan 2016] finds the probability that one or two specified points occur in a random permutation avoiding 312; and the work of several authors [Deutsch et al. 2002/03; Firro et al. 2007] determines the lengths of the longest monotone and alternating subsequences in a random permutation avoiding a single pattern of size 3. We also consider uniformly random 321-avoiding and 231-avoiding permutations in our work, but the statistics we are concerned with arise from the game of best choice. In some sense, our results refine the question of where a uniformly random pattern-avoiding permutation achieves its maximum because in our problem we want to transition so as to capture the maximum value. We also consider asymptotics for both of our models, thus obtaining a “limit-strategy,” just as in the classical game.

In addition, Wilf [1995] has collected some results on distributions of left-to-right maxima and Prodinger [2002] has studied these under a geometric random model. Although we phrase our results in terms of the game of best choice, they may also be viewed as an extension of the literature on distributions of left-to-right maxima to subsets of pattern-avoiding permutations.

3. Raising the bar

An *extension* of a permutation $p = p_1 p_2 \cdots p_{N-1}$ is the result of inserting value N into one of the N positions before, between, or after entries in p .

Lemma 3.1. *Let p be a 231-permutation of size $N - 1$ and $0 \leq k \leq N - 1$. Then there exists a unique extension of p that is k -winnable for N .*

Proof. Fix N and k . Let $p_1 p_2 \cdots p_k | p_{k+1} \cdots p_{N-1}$ be a 231-avoiding permutation of size $N - 1$, with $p_m = \max\{p_1, p_2, \dots, p_k\}$.

Define p_w to be the leftmost value greater than p_m among $\{p_{k+1}, p_{k+2}, \dots, p_{N-1}\}$, and let q be the result of inserting N into the position directly prior to p_w (or into the last position if p_w does not exist). So we have

$$q = p_1 p_2 \cdots p_m \cdots p_k | p_{k+1} \cdots p_{w-1} N p_w \cdots p_{N-1}.$$

We claim that q is the unique 231-avoiding k -winnable extension of p . To see this, observe that:

- By construction, all elements of $\{p_{k+1}, \dots, p_{w-1}\}$ are less than p_m , so q is k -winnable.
- We began with a 231-avoiding permutation p . If q contains 231, the value N must play the role of “3”. Therefore, it suffices to show that all of the values lying to the left of N are less than all values lying to the right of N . By construction, $p_m = \max\{p_1, p_2, \dots, p_{w-1}\}$ and $p_m < p_w$. If there exists some element $y < p_m$ among the entries $p_{w+1}, p_{w+2}, \dots, p_{N-1}$ then (p_m, p_w, y) forms a 231-instance, contradicting that p is 231-avoiding. Hence, no such y exists and q is 231-avoiding.
- If the extension q were not unique, we would have two positions L_1 and L_2 , say, where N could be inserted to the right of p_k to produce distinct k -winnable permutations of size N . In particular, there must exist at least one element p_v between L_1 and L_2 . But the previous paragraph shows that we would require $p_m < p_v$ for the extension q using L_1 to be 231-avoiding, so the extension using L_2 is not k -winnable, a contradiction. Hence, the extension is unique. \square

It is well known that the Catalan numbers

$$C_N = \frac{1}{N+1} \binom{2N}{N}$$

count the number of 231-avoiding permutations of size N ; see, e.g., [Bóna 2012]. Hence, we obtain the following result.

Corollary 3.2. *There are exactly C_{N-1} permutations of size N that are 231-avoiding and k -winnable.*

Proof. For fixed k , the set of 231-avoiding permutations of size $N-1$ are in bijection with the set of 231-avoiding k -winnable permutations of size N by Lemma 3.1. \square

Notice the curious consequence that *it does not matter which positional strategy we use*: for fixed N , the probability of selecting the best candidate is the same for all k . From the explicit formula, it is straightforward to work out the asymptotic probability of success

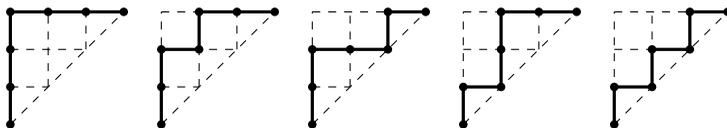
$$\lim_{N \rightarrow \infty} \frac{C_{N-1}}{C_N} = \frac{1}{4}.$$

4. Avoiding disappointment

Next, we consider positional strategies for the 321-avoiding interview rank orders. Recall that a permutation is k -winnable if and only if k lies between its last two left-to-right maxima. Hence, we study the distribution of left-to-right maxima in 321-avoiding permutations. For this, we make use of *Dyck paths*. These may be

viewed as paths in the Cartesian plane from $(0, 0)$ to (N, N) , consisting of $(0, 1)$ steps (i.e., north) and $(1, 0)$ steps (i.e., east), staying above the line $y = x$. The *northeast corners* in a Dyck path consist of a north step immediately followed by an east step. We label each northeast corner by the column and height at the end of its east step.

Example 4.1. The Dyck paths for $N = 3$ are shown below:



Their sets of northeast corners are

$$\{(1, 3)\}, \{(1, 2), (2, 3)\}, \{(1, 2), (3, 3)\}, \{(1, 1), (2, 3)\}, \{(1, 1), (2, 2), (3, 3)\}$$

respectively.

Lemma 4.2. *The possible sets $\{p_{i_1}, p_{i_2}, \dots, p_{i_m}\}$ of values and positions of left-to-right maxima arising from the various permutations of N are in bijection with the sets of northeast corners*

$$\{(i_j, p_{i_j}) : j = 1, \dots, m\}$$

of Dyck paths of size N .

Proof. The defining property for a Dyck path is that at each step along the path, the number of east steps taken so far is less than or equal to the number of north steps taken so far. Equivalently, we may consider paths whose northeast corners satisfy the following two conditions:

- There is always a northeast corner in the first column.
- Whenever we add a northeast corner corresponding to p_{i_j} , we take at most $p_{i_j} - i_j$ east steps until we reach the next column with a northeast corner.

But this is precisely equivalent to the conditions that define sets of left-to-right maxima in a permutation:

- The first position is always a left-to-right maximum.
- Whenever we add a left-to-right maximum corresponding to p_{i_j} , we have (by definition) at most $p_{i_j} - i_j$ complementary values that are smaller than p_{i_j} and have not yet been used. Hence, there are at most $p_{i_j} - i_j$ entries until we reach the next left-to-right maximum.

Given a Dyck path representing a set of left-to-right maxima, we can produce a canonical permutation p that realizes this set of left-to-right maxima as follows: Place each p_{i_j} into position i_j and then fill the complementary positions with

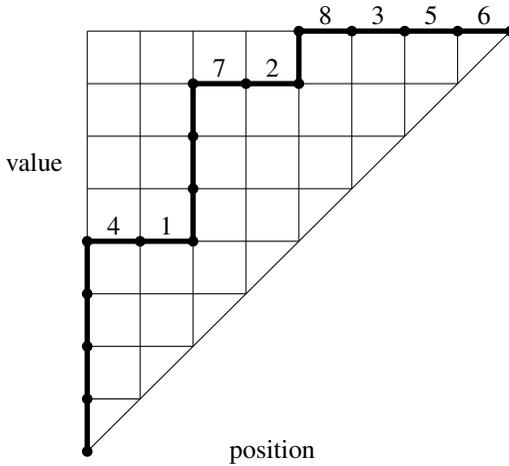


Figure 1. Completing the set of left-to-right maxima $\{p_1=4, p_3=7, p_5=8\}$.

the complementary values $\{1, 2, \dots, N\} \setminus \{p_{i_1}, \dots, p_{i_m}\}$ arranged increasingly. In terms of the Dyck path, we can label each northeast corner by the value of its corresponding left-to-right maximum, and then label the remaining horizontal edges with the complementary values, arranged increasingly as we read north and east along the path. Thus, the label for column i of the Dyck path gives the value for the i -th position of the permutation. \square

As an example in $N = 8$, if $p_1 = 4, p_3 = 7$, and $p_5 = 8$ are the p_{i_j} , we obtain $p = 41728356$; this is illustrated in [Figure 1](#).

Recall that the Catalan numbers C_N count 321-avoiding permutations of size N , and also count the number of Dyck paths of size N ; see, e.g., [\[Bóna 2012\]](#). Hence, we obtain the following result.

Corollary 4.3. *A 321-avoiding permutation p of size N is uniquely determined by the values and positions of its left-to-right maxima.*

Proof. The construction in the previous proof produces C_N distinct permutations of size N that have the structure of two increasing sequences shuffled together (namely, the sequence of left-to-right maxima, and the sequence of complementary values). Hence, the permutations constructed from Dyck paths in the previous result are all 321-avoiding. Since there are Catalan-many of each, there must be exactly one 321-avoiding permutation for each Dyck path. \square

Definition 4.4. For $1 \leq i \leq N - 1$ define $T_i(N)$ to be the total number of partial Dyck paths from $(0, 0)$ to $(N - 1 - i, N - 1)$, and define $S_i(N)$ to be the number of Dyck paths from $(0, 0)$ to (N, N) where column $N - i$ lies weakly right of the next-to-last northeast corner and strictly left of the last northeast corner in the path.

$N \setminus k$	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
2											1
3										3	2
4									6	8	5
5								10	20	23	14
6							15	40	65	70	42
7						21	70	145	214	222	132
8					28	112	280	514	717	726	429
9				36	168	490	1064	1817	2442	2431	1430
10			45	240	798	1988	3962	6446	8437	8294	4862
11		55	330	1230	3444	7784	14636	22997	29510	28730	16796
12	66	440	1815	5628	14154	29924	53937	82550	104312	100776	58786

Table 1. Number of k -winnable 321-avoiding permutations of N .

By [Corollary 4.3](#), $S_i(N)$ is the number of $(N - i)$ -winnable permutations of N . For example, the path in [Figure 1](#) would be counted in $S_i(N)$ for $N - i \in \{3, 4\}$ because the last two northeast corners occur in columns 3 and 5, respectively. Some initial values are given in [Table 1](#). If we divide by the N -th Catalan number we obtain the probability of success for the corresponding $(N - i)$ -positional strategy. These are illustrated in [Table 2](#). It turns out that the $T_i(N)$ are Catalan triangle entries at $(N - 1, i)$, namely

$$T_i(N) = \frac{i+1}{N} \binom{2(N-1)-i}{N-1},$$

but we do not use this in our development.

Now, define an operation Δ that acts on a function of N by replacing N with $N - 1$. That is, $\Delta f(N) = f(N - 1)$. We prefer to use this operator, with the argument N suppressed, as a notational convenience for our formulas and figures (although all of our results can be obtained without it). We next prove recurrences for the S_i and T_i that will facilitate their computation.

Theorem 4.5. *We have*

$$T_i = T_{i-1} - \Delta T_{i-2},$$

with $T_1 = C_{N-1}$ and $T_2 = C_{N-1} - C_{N-2}$, and

$$S_i = i T_i + \Delta S_{i-1},$$

with $S_1 = C_{N-1}$.

Proof. See [Figure 2](#) for a schematic illustrating these recurrences.

The recurrence for T follows because each path counted by $T_{i-1}(N)$ must end with a vertical step or a horizontal step; these are counted by $\Delta T_{i-2}(N) = T_{i-2}(N - 1)$ and $T_i(N)$, respectively.

$N \setminus k$	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
2											50.0
3										60.0	40.0
4									42.8	57.1	35.7
5								23.8	47.6	54.7	33.3
6							11.3	30.3	49.2	53.0	31.8
7						4.89	16.3	33.7	49.8	51.7	30.7
8				1.95	7.83	19.5	35.9	50.1	50.7	30.0	
9			0.74	3.45	10.0	21.8	37.3	50.2	50.0	29.4	
10		0.26	1.42	4.75	11.8	23.5	38.3	50.2	49.3	28.9	
11		0.09	0.56	2.09	5.85	13.2	24.8	39.1	50.1	48.8	28.5
12	0.03	0.21	0.87	2.7	6.8	14.3	25.9	39.6	50.1	48.4	28.2
13	0.07	0.34	1.18	3.26	7.61	15.3	26.7	40.1	50.0	48.0	28.0
14	0.13	0.49	1.47	3.76	8.31	16.1	27.4	40.4	50.0	47.7	27.7
15	0.19	0.63	1.75	4.21	8.92	16.8	28.0	40.7	49.9	47.5	27.5
16	0.26	0.78	2.01	4.61	9.46	17.3	28.5	41.0	49.9	47.2	27.4
17	0.33	0.92	2.26	4.98	9.93	17.8	28.9	41.2	49.8	47.0	27.2
18	0.4	1.05	2.48	5.31	10.3	18.3	29.2	41.3	49.7	46.8	27.1
⋮											
10^5	2.73	4.49	7.22	11.3	17.1	24.9	34.2	43.3	48.4	43.7	25.0

Table 2. Percentage of k -winnable 321-avoiding permutations of N .

The recurrence for S follows because each path counted by $S_i(N)$ passes through column $N - i$ at level $N - 1$ or passes through column $N - i$ below level $N - 1$. The first set of paths is counted by $iT_i(N)$ because any path ending at $(N - 1 - i, N - 1)$ can be extended in i ways depending on which of the columns $N - i, N - i + 2, \dots, N - 1$ is used for the last vertical step. The second set of paths is counted by $\Delta S_{i-1}(N) = S_{i-1}(N - 1)$ because we can bijectively extend any path passing the required column and ending at $(N - 1, N - 1)$ to end at (N, N) instead by inserting one more pair of vertical/horizontal steps at the last northeast corner. \square

Using this theorem, we may write each S_i and T_i as a linear combination of Catalan numbers. On the one hand, applying Δ to S_i , say, simply restricts the Dyck paths we are counting to end at $(N - 1, N - 1)$ instead of (N, N) . Algebraically, applying Δ replaces each Catalan number in the linear combination with the previous Catalan number.

Example 4.6. Applying the recurrences from [Theorem 4.5](#), we have

$$T_3 = (C_{n-1} - C_{n-2}) - \Delta(C_{n-1}) = C_{n-1} - 2C_{n-2},$$

$$T_4 = (C_{n-1} - 2C_{n-2}) - \Delta(C_{n-1} - C_{n-2}) = C_{n-1} - 3C_{n-2} + C_{n-3},$$

$$T_5 = (C_{n-1} - 3C_{n-2} + C_{n-3}) - \Delta(C_{n-1} - 2C_{n-2}) = C_{n-1} - 4C_{n-2} + 3C_{n-3}$$

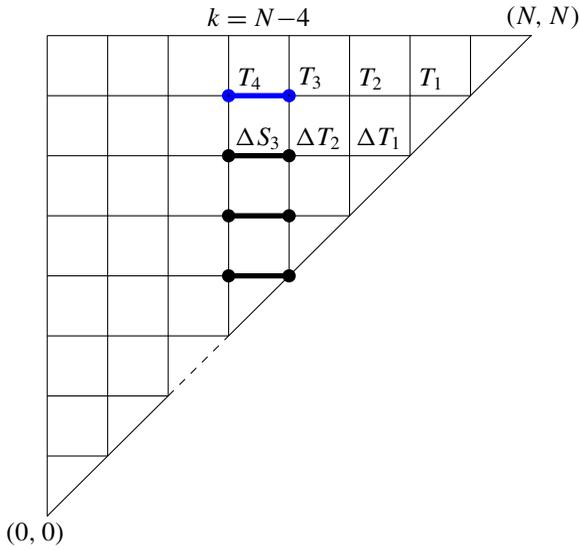


Figure 2. Schematic for path recurrences.

and

$$S_2 = 2(C_{n-1} - C_{n-2}) + \Delta(C_{n-1}) = 2C_{n-1} - C_{n-2},$$

$$S_3 = 3(C_{n-1} - 2C_{n-2}) + \Delta(2C_{n-1} - C_{n-2}) = 3C_{n-1} - 4C_{n-2} - C_{n-3},$$

$$S_4 = 4(C_{n-1} - 3C_{n-2} + C_{n-3}) + \Delta(3C_{n-1} - 4C_{n-2} - C_{n-3}) = 4C_{n-1} - 9C_{n-2} - C_{n-4},$$

$$S_5 = 5(C_{n-1} - 4C_{n-2} + 3C_{n-3}) + \Delta(4C_{n-1} - 9C_{n-2} - C_{n-4}) \\ = 5C_{n-1} - 16C_{n-2} + 6C_{n-3} - C_{n-5}.$$

Lemma 4.7. Let $i \leq N - 5$ and X_i be a linear combination of the Catalan numbers $C_{N-1}, C_{N-2}, \dots, C_{N-i}$. Then,

$$\frac{1}{4} \frac{X_i}{C_N} < \frac{\Delta X_i}{C_N} \leq \frac{1}{3} \frac{X_i}{C_N}.$$

Proof. Observe that

$$\frac{1}{4} < \frac{C_{N-1}}{C_N} \leq \frac{1}{3}$$

for all $N \geq 5$. Since

$$\frac{\Delta C_{N-i}}{C_N} = \frac{C_{N-i-1}}{C_N} = \frac{C_{N-i-1}}{C_{N-i}} \frac{C_{N-i}}{C_N},$$

we have

$$\frac{1}{4} \frac{C_{N-i}}{C_N} < \frac{\Delta C_{N-i}}{C_N} \leq \frac{1}{3} \frac{C_{N-i}}{C_N}$$

for all $N - i \geq 5$, and the result follows by linearity. □

Lemma 4.8. *For all $i \leq N - 5$, we have*

$$\frac{T_i}{C_N} \leq \frac{1}{3} \left(\frac{3}{4}\right)^{i-1}.$$

Proof. It is straightforward to verify that the result holds for $i = 1$ and $i = 2$. Suppose the result holds for $i - 1$. Then,

$$\frac{T_i}{C_N} = \frac{T_{i-1}}{C_N} - \frac{\Delta T_{i-2}}{C_N} < \frac{T_{i-1}}{C_N} - \frac{1}{4} \frac{T_{i-2}}{C_N}$$

by [Lemma 4.7](#). From their definition in terms of lattice paths, it is also clear that the T_i are decreasing in i (for each fixed N). Hence,

$$\frac{T_{i-1}}{C_N} - \frac{1}{4} \frac{T_{i-2}}{C_N} \leq \frac{T_{i-1}}{C_N} - \frac{1}{4} \frac{T_{i-1}}{C_N} = \frac{3}{4} \frac{T_{i-1}}{C_N} \leq \frac{1}{3} \left(\frac{3}{4}\right)^{i-1}$$

by induction. □

Theorem 4.9. *We have*

$$\frac{S_3}{C_N} > \frac{S_i}{C_N}$$

for all $N \geq 9$ and all $i > 3$.

Proof. We have

$$\frac{S_i}{C_N} = \frac{iT_i + \Delta S_{i-1}}{C_N} \leq \frac{i}{3} \left(\frac{3}{4}\right)^{i-1} + \frac{1}{3} \frac{S_{i-1}}{C_N}.$$

An exercise using calculus proves $\frac{i}{3} \left(\frac{3}{4}\right)^{i-1}$ is decreasing once $i > -1/\ln(\frac{3}{4})$ (which is between 3 and 4) and $\frac{i}{3} \left(\frac{3}{4}\right)^{i-1}$ is less than $\frac{1}{4}$ for all $i \geq 11$. Consequently, once $S_i/C_N < \frac{3}{8}$, it remains so as i increases for all $i \geq 11$.

In fact, using the linear combinations of Catalan numbers obtained from [Theorem 4.5](#) as in [Example 4.6](#), we can verify that $S_i/C_N < \frac{3}{8}$ for all $5 \leq i \leq 11$ as illustrated in [Table 2](#). More precisely, when we express S_i/C_N as a linear combination of ratios of Catalan numbers, the limiting value as $N \rightarrow \infty$ can be obtained by plugging in powers of $\frac{1}{4}$ for each ratio of Catalan numbers; as these limits are each smaller than $\frac{3}{8}$, we reduce to a finite computation. In detail, we use the bounds

$$0.25^j < \frac{C_{N-j}}{C_N} < 0.254^j$$

for $N > 95 + j$ to verify that $S_i/C_N < \frac{3}{8}$ for each of the linear combinations $i = 5, 6, \dots, 11$ (and check remaining finite cases for N manually).

Thus, the optimal value of S_i/C_N must occur in $i \leq 4$ for all N . Using the formulas from [Example 4.6](#) again, we then find that S_1/C_N is optimal for $N = 2$, that S_2/C_N is optimal for $3 \leq N \leq 8$, and that S_3/C_N is optimal for all $N \geq 9$. □

Corollary 4.10. *The optimal k -positional strategy for the game of best choice restricted to the 321-avoiding interview rank orders is*

$$k = \begin{cases} N - 1 & \text{if } N = 2, \\ N - 2 & \text{if } 3 \leq N \leq 8, \\ N - 3 & \text{otherwise.} \end{cases}$$

The asymptotic probability of success is

$$\lim_{N \rightarrow \infty} \frac{3C_{N-1} - 4C_{N-2} - C_{N-3}}{C_N} = \frac{31}{64} = 0.484375.$$

Using André’s reflection method or a straightforward induction argument, one can show that the number of partial Dyck paths (i.e., lying above the line $y = x$) from $(0, 0)$ to (a, b) (where $a < b$) is given by the formula

$$C_{(a,b)} = \binom{a+b}{a} \frac{b-a+1}{b+1}.$$

Using this, we can also give a direct count of the Dyck paths for which column k lies between the last two northeast corners of the path.

Theorem 4.11. *The probability that a 321-avoiding permutation of length N is k -winnable is*

$$\frac{1}{C_N} \sum_{i=1}^{N-k} \binom{(k-1)+(N-i)}{k-1} \frac{(N-k-i+2)}{(N-i)+1} (N-k-i+1).$$

Proof. Set $a = k - 1$, and let b range over $k, k + 1, k + 2, \dots, N - 1$. Once the path passes through (a, b) , there are $b - k + 1$ ways to complete it so that it is k -winnable. □

5. Conclusions

It seems fair to say that these results are somewhat surprising and further investigation is warranted. The “bar-raising” model has a robust strategy but only allows a 25% success rate. The optimal strategy in the “disappointment-free” model reviews and rejects most of the applicants yet has a success rate that is close to 50%. Remarkably, these are not mutually exclusive and the $k = N - 3$ positional strategy is asymptotically optimal in both models simultaneously.

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