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# Patterns in colored circular permutations

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Pattern containment and avoidance have been extensively studied in permutations. Recently, analogous questions have been examined for colored permutations and circular permutations. In this note, we explore these problems in colored circular permutations. We present some interesting observations, some of which are direct generalizations of previously established results. We also raise some questions and propose directions for future study.

## 1. Background

Patterns are essentially subpermutations of a bigger permutation. For two permutations $\pi$ and $\tau$ of lengths $n$ and $k$ with $n \geq k$, we say that $\pi$ *contains* $\tau$ *as a pattern* if there is a subsequence of entries of $\pi$, $(\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \ldots, \pi_{i_k})$, which is order isomorphic to $\tau$; i.e., $\pi_{i_s} \leq \pi_{i_t}$ if and only if $\tau_s \leq \tau_t$. Such a subsequence is called an *occurrence* of $\tau$ in $\pi$. If no occurrence of $\tau$ is present in $\pi$, we say that $\pi$ *avoids* $\tau$.

Most of the earlier work on patterns concerns *pattern avoidance*; see [Bóna 2012] for a nice introduction. A more comprehensive study of pattern containment was first proposed by H. Wilf in 1992 [Liendo 2012]. There are two natural questions one might ask regarding pattern containment. First, what is the shortest permutation that contains every element in some set of permutations? Second, for a given pattern, in what permutation does this pattern occur the most? The former deals with *superpatterns*, whereas the latter concerns *pattern packing*.

***Superpatterns.*** For a set $\mathcal{P}$ of permutations we say that a permutation $\pi$ is a $\mathcal{P}$-superpattern if it contains at least one occurrence of every $\tau \in \mathcal{P}$. We also define

$$\mathrm{sp}(\mathcal{P}) = \min\{n : \text{there is a } P\text{-superpattern of length } n\}$$

and $\mathrm{sp}(k) = \mathrm{sp}(\mathcal{P})$ when $\mathcal{P}$ is the set of all permutations of length $k$.

For results on the bounds of $\mathrm{sp}(k)$, see [Arratia 1999; Eriksson et al. 2007; Miller 2009]. Bounds of $\mathrm{sp}(P)$ have also been studied for layered permutations [Gray

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2015], 321-avoiding permutations [Bannister et al. 2014], $m$-colored permutations [Gray and Wang 2016], and words [Burstein et al. 2002/03].

***Pattern packing.*** Letting $f(\pi, \tau)$ be the number of occurrences of $\tau$ in $\pi$, we define

$$g(n, \tau) = \max\{f(\sigma, \tau) : \sigma \text{ is a permutation of length } n\}.$$

and the *packing density* of $\tau$ as

$$\delta(\tau) = \lim_{n\to\infty} \frac{g(n, \tau)}{\binom{n}{k}}.$$

A permutation $\pi$ (of length $n$) with $f(\pi, \tau) = g(n, \tau)$ is called $\tau$*-optimal*.

For packing densities of length-3 and length-4 patterns, see [Albert et al. 2002; Price 1997; Stromquist 1993]. There are three length-4 patterns whose packing densities remain open, as are any longer nonlayered patterns.

***Pattern avoidance.*** Pattern avoidance has been well-studied for permutations; see [Bóna 2012] for details. In the case of colored permutations, [Mansour 2001] provides a formula for the number of permutations avoiding all length-2 permutations whose entries are colorable in $r$ ways. For circular permutations, [Callan 2002] counts the number of circular permutations avoiding 1324, 1342, and 1234. Both topics are relatively new, and there are still many open questions.

***Our contribution.*** There are two natural variations of permutations, colored permutations and circular permutations, where the first one assigns colors to each entry and the second arranges entries around a circle. In colored permutations, superpatterns [Gray and Wang 2016], pattern packing [Just and Wang 2016], and pattern avoidance [Mansour 2001] have been considered. Noncolored pattern containment [Gray et al. 2017] and pattern avoidance [Callan 2002] have been studied for circular patterns. In this paper, we will consider the combination of these two variations, the colored circular permutations. First, we will introduce the necessary terminology and notation in Section 2. We then discuss "supercolored circular permutations" in Section 3, where we point out that many of the results in [Gray and Wang 2016] can be directly generalized to the colored circular permutations. In Section 4, we discuss pattern packing in colored circular permutations, including some generalizations of results in [Just and Wang 2016]. Lastly, in Section 5 we consider pattern avoidance in colored circular permutations. We conclude our work by commenting on the many remaining problems for future work in Section 6.

## 2. Terminologies in colored and circular permutations

We start with some formal terminologies and notations for colored permutations and patterns.

**Definition 2.1.** Let $k$ and $m$ be any positive integers. An *$m$-colored permutation of length $k$* is any permutation of length $k$ where each entry is colored one of $m$ given colors; we allow distinct entries of the permutation to be colored differently. We denote the set of all permutations of length $k$ in $m$ colors by $\mathcal{S}_{k,m}$.

In the case that there are only two or three colors, we will color the entries of a permutation “red”, “green”, or “blue”; thus, we may have the colored permutation $2_r 1_b 3_r$, which denotes the permutation 213 whose first and third entries are colored “red” and whose second entry is colored “blue”. If more than three colors are allowed, we will just label the colors with natural numbers; hence, the colored permutation $1_1 3_4 4_1 5_3 2_2$ is the permutation 13452 whose first and third entries are colored 1, fifth entry is colored 2, fourth entry is colored 3, and second entry is colored 4.

**Definition 2.2.** Let $k$ and $m$ be any positive integers. A *monochromatic* $m$-colored permutation of length $k$ is any $m$-colored permutation for which every entry is colored the same color. We denote the set of all monochromatic $m$-colored permutations of length $k$ by $\mathcal{M}_{k,m}$.

**Definition 2.3.** Let $k$ and $m$ be any positive integers. A *nonmonochromatic* $m$-colored permutation of length $k$ is any $m$-colored permutation for which there exist at least two distinct entries that are colored differently. We denote the set of all nonmonochromatic $m$-colored permutations of length $k$ by $\mathcal{N}_{k,m}$.

The union of $\mathcal{N}_{k,m}$ and $\mathcal{M}_{k,m}$ is $\mathcal{S}_{k,m}$, the set of all $m$-colored permutations of length $k$. For example, $1_r 2_b 3_b$ is nonmonochromatic since the first entry and second entry are colored differently, while $1_r 2_r 3_r$ and $1_b 2_b 3_b$ are both monochromatic. For comparison, we list $\mathcal{S}_{2,2}$, $\mathcal{N}_{2,2}$, and $\mathcal{M}_{2,2}$ below:

$$\begin{aligned}
\mathcal{S}_{2,2} &= \{1_r 2_r, 1_r 2_b, 1_b 2_r, 1_b 2_b, 2_r 1_r, 2_r 1_b, 2_b 1_r, 2_b 1_b\},\\
\mathcal{N}_{2,2} &= \{1_r 2_b, 1_b 2_r, 2_r 1_b, 2_b 1_r\},\\
\mathcal{M}_{2,2} &= \{1_r 2_r, 1_b 2_b, 2_r 1_r, 2_b 1_b\}.
\end{aligned}$$

**Definition 2.4.** For colored permutations $p$ and $q$ we say that $p$ contains $q$ as a colored pattern if there is some subsequence of $p$, say $P$, which satisfies the following two conditions:

- The $i$-th entry of $P$ is the same color as the $i$-th entry of $q$ for all $i$.
- $P$ is order isomorphic to $q$.

If there is no such $P$ satisfying both conditions, we say that *$p$ avoids $q$* as a colored pattern.

We will usually drop the phrase “as a colored pattern” and just say “$p$ contains $q$” when it is obvious that we are dealing with colored permutations. For instance, if

$p = 1_r 3_b 2_r$ and $q = 2_b 1_r$, we see that $p$ contains $q$ since the subsequence $(3_b, 2_r)$ of $p$ satisfies both of the conditions above. However, if $q = 2_r 1_b$ then $p$ avoids $q$ since there is no subsequence of $p$ simultaneously satisfying both conditions.

Similar to the noncolored case, for a collection $\mathcal{P}$ of colored permutations we define the $\mathcal{P}$-superpattern and $\mathrm{sp}(\mathcal{P})$ accordingly. Note that the permutation $p = 1_r 2_b 6_r 5_b 4_r 3_b$ contains every colored permutation in $\mathcal{S}_{2,2}$. Hence, $p$ is an $\mathcal{S}_{2,2}$-superpattern. Brute force shows that there is no shorter $\mathcal{S}_{2,2}$-superpattern; therefore $\mathrm{sp}(\mathcal{S}_{2,2}) = 6$.

Next we formalize the concept of pattern containment/avoidance in circular permutations. Note that the following definition also applies to colored permutations.

**Definition 2.5.** Let $p = p_1 p_2 \cdots p_n$ be a permutation of length $n$. The *circular shift* of $p$, denoted $S(p)$, is given by

$$S(p) = p_n p_1 p_2 \cdots p_{n-1}.$$

If we take a permutation, $\pi$, and wrap its entries clockwise around a circle, equally spread out within one revolution, then we have created a circular permutation, $\pi_c$. We say that $\pi_c = \tau_c$ if $\tau$ is just a cyclic shift of $\pi$, i.e., $S^i(\tau) = \pi$ for some $i$.

**Definition 2.6.** For colored permutations $p$ and $q$ we say that $p$ contains $q$ *circularly* if $p$ contains $S^i(q)$ as a colored pattern for some nonnegative integer $i$.

**Definition 2.7.** Let $\mathcal{P}$ be any collection of permutations. A *circular $\mathcal{P}$-superpattern* is a permutation which contains every $p \in \mathcal{P}$ as a circular pattern. We let $\mathrm{sp}_c(\mathcal{P})$ denote the length of the shortest circular $\mathcal{P}$-superpattern. When $\mathcal{P}$ is the set of all circular patterns of length $k$ we simply write $\mathrm{sp}_c(k)$.

A useful concept in the study of pattern packing in colored permutations will be "colored blocks", which we define below.

**Definition 2.8.** In a colored permutation $\pi$, a *colored block* is a maximal monochromatic segment $\pi_i^{(a)}$ in which every entry in this segment has color $a$ and every entry not in this segment is either larger or smaller than each entry in $\pi_i^{(a)}$.

For example, the permutation $\pi = 1_r 2_r 6_b 5_b 3_b 4_r$ has four colored blocks. From left to right, they are $\pi_1^{(r)} = 1_r 2_r$, $\pi_2^{(b)} = 6_b 5_b$, $\pi_3^{(b)} = 3_b$, $\pi_4^{(r)} = 4_r$. Indeed every colored permutation has a unique decomposition into colored blocks: Given a colored permutation, it can first be decomposed into maximal monochromatic subsequences and it is easy to see that there is a unique way to do this. Within each monochromatic subsequence there is a unique way to separate the entries according to their numerical values.

When comparing the numerical values between different blocks, we say that $\pi_i^{(r)} < \pi_j^{(b)}$ when all entries of $\pi_i^{(r)}$ are less than all entries of $\pi_j^{(b)}$. It is easy to see that this concept generalizes naturally to the circular case.

For colored patterns $\tau$ and permutations $\pi$ we define $f_c(\pi, \tau)$ to be the number of occurrences of $\tau$ in $\pi$ wrapped around a circle; i.e.,

$$f_c(\pi, \tau) = f(\pi, \tau) + f(\pi, S(\tau)) + f(\pi, S^2(\tau)) + \cdots + f(\pi, S^{k-1}(\tau)).$$

Then, similar to before,

$$g_c(n, \tau) = \max\{f_c(\sigma, \tau) : \sigma \text{ is a permutation of length } n\}.$$

If $\pi$ is of length $n$ and $f_c(\pi, \tau) = g_c(n, \tau)$, then we say that $\pi$ is *circular $\tau$-optimal.*

**Definition 2.9.** Let $\tau$ be a colored permutation of length $k$. The *circular packing density* of $\tau$, denoted by $\delta_c(\tau)$, is defined by

$$\delta_c(\tau) = \lim_{n\to\infty} \frac{g_c(n, \tau)}{\binom{n}{k}}.$$

## 3. Superpatterns

In this section we consider questions related to superpatterns in colored circular permutations. We note that some of the results in this section are direct generalizations from those in [Gray and Wang 2016]. For this reason some details will be omitted.

**Theorem 3.1.** *For any positive integers $k$ and $m$, we have that*

$$\operatorname{sp}_c(\mathcal{S}_{k,m}) = m \operatorname{sp}_c(k).$$

*Proof.* Let $p'$ be a circular $\mathcal{S}_{k,m}$-superpattern and $p'_i$ be the longest monochromatic subsequence in $p'$ in color $i$. It follows that $p'$ is a circular $k$-superpattern and consequently $|p'_i| \geq \operatorname{sp}_c(k)$ for any $1 \leq i \leq m$. Hence

$$|p'| = \sum_{i=1}^{m} |p'_i| \geq m \operatorname{sp}_c(k).$$

Now, let $p$ be a circular permutation of length $\operatorname{sp}_c(k)$ that contains all noncolored patterns of length $k$. Consider the $m$-colored circular permutation $p''$, constructed from $p$ by replacing each $1 \leq j \leq \operatorname{sp}_c(k)$ in $p$ with the sequence

$$s_j = [m(j-1)+1]_1 [m(j-1)+2]_2 \cdots [m(j-1)+m]_m.$$

It is easy to see that

$$|p''| = m|p| = m \operatorname{sp}_c(k)$$

and that $p''$is a $\mathcal{S}_{k,m}$-superpattern. Thus,

$$\operatorname{sp}_c(\mathcal{S}_{k,m}) \leq |p''| = m \operatorname{sp}_c(k). \qquad \square$$

With Theorem 3.1, we can use previously established results [Gray et al. 2017] on $\operatorname{sp}_c(k)$ to bound $\operatorname{sp}_c(\mathcal{S}_{k,m})$.

**Corollary 3.2.** *For positive integers $k$ and $m$ we have*

$$\mathrm{sp}_c(\mathcal{S}_{k,m}) = m\,\mathrm{sp}_c(k) \geq mg(k)\frac{k^2}{e^2},$$

*where $g(k) \to 1$ as $k \to \infty$, and*

$$\mathrm{sp}_c(\mathcal{S}_{k,m}) = m\,\mathrm{sp}_c(k) \leq m(\mathrm{sp}(k-1)+1) \leq m\big(\tfrac{1}{2}k(k-1)+1\big).$$

*Consequently,*

$$mg(k)\frac{k^2}{e^2} \leq \mathrm{sp}_c(\mathcal{S}_{k,m}) \leq m\big(\tfrac{1}{2}k(k-1)+1\big),$$

*where $g(k) \to 1$ as $k \to \infty$.*

As expected, the bounds for $\mathrm{sp}_c(\mathcal{S}_{k,m})$ are simply $m$ times the bounds for $\mathrm{sp}_c(k)$. Next, we restrict our attention to only monochromatic or nonmonochromatic patterns. First we note the following facts on the sizes of $\mathcal{M}_{k,m}$ and $\mathcal{N}_{k,m}$:

- $|\mathcal{S}_{k,m}| = m^k k!\,.$
- $|\mathcal{M}_{k,m}| = mk!\,.$
- $|\mathcal{N}_{k,m}| = |\mathcal{S}_{k,m}| - |\mathcal{M}_{k,m}| = (m^{k-1}-1)|\mathcal{M}_{k,m}|.$

We now establish a lower bound for $\mathrm{sp}_c(\mathcal{N}_{k,m})$.

**Theorem 3.3.** *For positive integers $k$ and $m$,*

$$\mathrm{sp}_c(\mathcal{N}_{k,m}) \geq m\,g(k,m)\frac{k^2}{e^2},$$

*where $g(k,m) \to 1$ as $k \to \infty$.*

*Proof.* Let $n = \mathrm{sp}_c(\mathcal{N}_{k,m})$, and our $\mathcal{N}_{k,m}$-superpattern of length $n$ must contain a circular shift of every permutation in $\mathcal{N}_{k,m}$. Note that at most $k$ such permutations can be circular shifts of each other; hence at least $|\mathcal{N}_{k,m}|/k$ permutations from $\mathcal{N}_{k,m}$ must be contained in the superpattern. Consequently

$$\binom{n}{k} \geq \frac{|\mathcal{N}_{k,m}|}{k} = \frac{(m^k-m)k!}{k} = (m^k-m)(k-1)!\,.$$

By the fact $n^k/k! \geq \binom{n}{k}$ and Stirling's approximation $k! \geq \sqrt{2\pi k}(k^k/e^k)$, we have

$$\frac{n^k}{k!} \geq (m^k-m)(k-1)!$$

and hence

$$\begin{aligned} n \geq \left((m^k-m)\frac{(k!)^2}{k}\right)^{1/k} &\geq \left((m^k-m)2\pi\frac{k^{2k}}{e^{2k}}\right)^{1/k} \\ &= m\big((1-m^{-k+1})2\pi\big)^{1/k}\frac{k^2}{e^2} = mg(k,m)\frac{k^2}{e^2} \end{aligned}$$

with $g(k,m) = \big((1-m^{-k+1})2\pi\big)^{1/k} \to 1$ as $k \to \infty$. □

It is interesting to note that this lower bound is similar to that found for $\mathrm{sp}_c(\mathcal{S}_{k,m})$ in Corollary 3.2. To bound $\mathrm{sp}_c(\mathcal{N}_{k,m})$ from above, first note that a circular $\mathcal{M}_{k,m}$-superpattern must have $m$ copies of a circular $k$-superpattern, one for each color. Then, we have

$$\mathrm{sp}_c(\mathcal{N}_{k,m}) \le \mathrm{sp}_c(\mathcal{S}_{k,m}) = m\,\mathrm{sp}_c(k) = \mathrm{sp}_c(\mathcal{M}_{k,m}).$$

Given the fact that $|\mathcal{N}_{k,m}| = (m^{k-1}-1)|\mathcal{M}_{k,m}|$, it is rather surprising that the shortest $\mathcal{N}_{k,m}$-superpattern is not longer than the shortest $\mathcal{M}_{k,m}$-superpattern. The following further analyzes the relationship between them.

**Theorem 3.4.** *For any positive integers $k \ge 2$ and $m$, we have*

$$\mathrm{sp}_c(\mathcal{M}_{k-1,m}) \le \mathrm{sp}_c(\mathcal{N}_{k,m}) \le \mathrm{sp}_c(\mathcal{M}_{k,m}).$$

*Proof.* The second inequality follows from the discussion above.

On the other hand, let $q$ be an $m$-colored pattern of length $k$ with all but one entry of color $i$. For some circular shift of $q$ to be contained in a circular $\mathcal{N}_{k,m}$-superpattern, a circular shift of the length-$(k-1)$ monochromatic pattern in color $i$ must be contained in the superpattern. Hence all length-$(k-1)$ monochromatic patterns (of any color) must occur in a circular $\mathcal{N}_{k,m}$-superpattern, and $\mathrm{sp}_c(\mathcal{M}_{k-1,m}) \le \mathrm{sp}_c(\mathcal{N}_{k,m})$. □

## 4. Pattern packing

Our results in this section mainly concern the characteristics of the optimal colored circular permutations when the pattern under consideration is described through colored blocks. Again, some of our results here are direct generalizations of those in noncircular case [Just and Wang 2016], for which reason we skip some details.

In the case of having only two colored blocks, we can see that a pattern must be of the form $\pi = \pi_1\pi_2$ with $\pi_1$ in red and $\pi_2$ in blue. We will assume, without loss of generality, that $\pi_1 < \pi_2$. In this case, we may simply say that the pattern is of the form $rb$ with $r < b$, and similarly for patterns with more colored blocks.

**Theorem 4.1.** *For a pattern $\rho$ with two colored blocks of the form $rb$ with $r < b$, there is an optimal circular permutation $\pi$ of the form $RB$ with $R < B$.*

*Proof.* Let $\pi$ be a $\rho$-optimal permutation of length $n$ with colored blocks $\pi_1\pi_2\cdots\pi_k$. We can assume without loss of generality that $\pi_1$ is red.

Now, let us take all the red blocks $\pi_{r_1}\pi_{r_2}\cdots\pi_{r_s}$ and blue blocks $\pi_{b_1}\pi_{b_2}\cdots\pi_{b_t}$, and form a new circular permutation $\pi' = \pi_{r_1}\cdots\pi_{r_s}\pi_{b_1}\cdots\pi_{b_t}$. It is easy to see that any occurrence of $\rho$ in $\pi$ is also in $\pi'$.

Next, since $\rho$ is of the form $rb$ with $r < b$, we claim that, in our optimal permutation $\pi'$, every red entry must be (numerically) less than every blue entry. Otherwise, one may always "rearrange" the numerical values so that the numerical ordering stays the same among entries of the same color, and so that all red entries are smaller than the blue ones. The resulting permutation can only contain more occurrences of $\rho$.

Consequently, all red blocks together simply form a single block in $\pi$ and so do the blue blocks. Our conclusion, then, follows. □

Next, we consider patterns with three colored blocks. Note that for circular patterns with three colored blocks and two colors, $rb_1b_2$ with $b_1 < r < b_2$ is the only case that we needed to investigate: With three colored blocks and two colors there are always one block with one color (say red) and two blocks with the other (say blue). One of the circular shifts of this pattern must be of the form $rb_1b_2$. By taking a circular shift of the reversed pattern (i.e., $rb_2b_1$) if necessary, we may also assume that $b_1 < r < b_2$.

**Theorem 4.2.** *For a pattern $\rho$ with three colored blocks of the form $rb_1b_2$ and $b_1 < r < b_2$, there is an optimal circular permutation $\pi$ of the same form.*

*Proof.* Let $\pi$ be a $\rho$-optimal circular permutation of length $n$. First, we will show that we can put all blue blocks in increasing order of their numerical values and next to each other. Let $\pi_{r_1}, \pi_{r_2}, \ldots, \pi_{r_s}$ be the red blocks of $\pi$.

Now, for an occurrence of $\rho$ in $\pi$, suppose $R_\rho$ (in $\pi$) is the part corresponding to $r$ (in $\rho$). Let $\pi_{b_{<R}}$ be the collection of all blue blocks (with numerical value) less than $R_\rho$, and let $\pi_{b_{>R_\rho}}$ be the set of all blue blocks greater than $R_\rho$. Then, any occurrence of $rb_1b_2$ with $r \sim R_\rho$ must have $b_1$ occurring in $\pi_{b_{<R_\rho}}$ and $b_2$ occurring in $\pi_{b_{>R_\rho}}$. The maximum number of such occurrences (i.e., the maximum possible contribution of $R_\rho$ to $f_c(\pi, \rho)$) is

$$f(\pi_{b_{<R_\rho}}, b_1) f(\pi_{b_{>R_\rho}}, b_2).$$

As far as the ordering of the blue blocks is concerned, arranging the blue blocks in increasing order achieves the above maximum. At this point it is also easy to see that putting blocks of the same color together will not reduce the number of occurrences of $\rho$. Denote such an optimal permutation by $\pi' = \pi_{r_1} \cdots \pi_{r_s} \pi_{b_1} \cdots \pi_{b_t}$, with $\pi_{b_i} < \pi_{b_{i+1}}$ for any $1 \le i \le t-1$.

Next, we show that all red entries form a single block, or equivalently, the numerical value of any red block is between those of $\pi_{b_{j_0-1}}$ and $\pi_{b_{j_0}}$ for some fixed $j_0$. Let $\pi_{b_{\ge j}}$ be the collection of blue blocks $\pi_{b_j}, \pi_{b_{j+1}}, \ldots, \pi_{b_t}$, and let $\pi_{b_{<j}}$ be the collection of blocks $\pi_{b_1}, \ldots, \pi_{b_j}$. Then, there must exist some $j_0$ that maximizes the occurrences of $b_1$ in $\pi_{b_{<j}}$ and $b_2$ in $\pi_{b_{\ge j}}$. In other words,

$$f(\pi_{b_{<j_0}}, b_1) f(\pi_{b_{\ge j_0}}, b_2) \ge f(\pi_{b_{<j}}, b_1) f(\pi_{b_{\ge j}}, b_2)$$

for any $1 < j \le t$. So,

$$f_c(\pi', \rho) \le f(\pi_{r_1} \cdots \pi_{r_s}, r) f(\pi_{b_{<j_0}}, b_1) f(\pi_{b_{\ge j_0}}, b_2)$$

with equality when $\pi_{b_{j_0-1}} < \pi_{r_i} < \pi_{b_{j_0}}$ for any $1 \le i \le s$. From this it follows that there are exactly one single red block and two blue blocks in $\pi = RB_1B_2$ with $B_1 < R < B_2$. $\square$

It remains to consider the case when we have three colored blocks in three different colors, i.e., the pattern $rbg$ with $r < b < g$.

**Theorem 4.3.** *For a pattern $\rho$ of the form $rbg$ with $r < b < g$, there is an optimal circular permutation $\pi$ of the same form.*

*Proof.* Let $\pi$ be a $\rho$-optimal circular permutation of length $n$ with $R'$, $B'$ and $G'$ being the collections of all red blocks (in their original order), blue blocks and green blocks respectively. An occurrence of $\rho$ in $\pi$ must consist of an occurrence of $r$ in $R'$, an occurrence of $b$ in $B'$, and an occurrence of $g$ in $G'$. Hence

$$f(\pi,\rho) \le f(R',r)f(B',b)f(G',g)$$

with equality if each of $R'$, $B'$ and $G'$ is a single block, arranged in this order, and $R' < B' < G'$. $\square$

To summarize the above observations, we have the following.

**Corollary 4.4.** *For any circular pattern with two or three colored blocks, there is a corresponding optimal circular permutation of the same form.*

**Remark 4.5.** After seeing the above results on patterns with two or three colored blocks, it is natural to guess that the same holds for patterns with more blocks. Consequently one can ask if there is always an optimal circular permutation of the same form as the pattern. We have not been able to prove either way.

On the other hand, considering $\rho = 1_r2_b$, it is not hard to check that $\pi = (1_r3_b2_r4_b)_c$ is an optimal length-4 circular permutation for $\rho$. Thus, there does exist an optimal permutation that is not of the form $RB$ with $R < B$. Evidence seems to suggest that this is the only such case.

## 5. Pattern avoidance

The numbers of $m$-colored (noncircular) permutations that avoid one or two 2-letter patterns were presented in [Mansour 2001], together with some discussion of the connection between pattern avoidance in noncolored permutations and colored permutations. In [Callan 2002], pattern avoidance in circular permutations was studied. It was pointed out that, when considered as circular permutations, none avoid any 2-letter patterns and the identity (reverse identity) is the only one avoiding the pattern 132 (123). In this section we extend this study to colored circular permutations, generalizing a little of both [Callan 2002] and [Mansour 2001].

***Avoiding a monochromatic length-2 pattern in $\mathcal{S}_{k,m}$.*** Without loss of generality, we may assume the monochromatic length-2 pattern is $1_1 2_1$. Then, to avoid such a pattern, our permutation $\pi$ in $\mathcal{S}_{k,m}$ can contain at most one entry of color 1. Consider two cases:

- There is no entry with color 1 in $\pi$. Then, there are a total of $(k-1)!$ noncolored circular permutations of length $k$, and each of the $k$ entries has $m-1$ choices of colors (i.e., the colors $2, 3, \ldots, m$); thus the number of such permutations is

$$(k-1)!\,(m-1)^k.$$

- There is exactly one entry with color 1. Out of the $k$ entries $1, 2, \ldots, k$ there are $k$ choices for this particular entry of color 1. There are still $(k-1)!$ ways to wrap the $k$ entries (regardless of their colors) around a circle. Now for each of the remaining $k-1$ entries that are not of color 1, there are $m-1$ choices of colors. Hence the number of such permutations is

$$k!\,(m-1)^{k-1}.$$

Consequently, we have the following.

**Theorem 5.1.** *The number of circular permutations in $\mathcal{S}_{k,m}$ that avoid a given monochromatic length-2 pattern is*

$$(k-1)!\,(m-1)^k + k!\,(m-1)^{k-1} = (k-1)!\,(m-1)^{k-1}(k+m).$$

***Avoiding nonmonochromatic length-2 pattern in $\mathcal{S}_{k,m}$.*** Again, without loss of generality, let us assume this pattern to be $1_1 2_2$. For a circular permutation $\pi$ in $\mathcal{S}_{k,m}$, let $E_1$ and $E_2$ be the sets of entries in $\pi$ that are colored 1 and 2 respectively. It is easy to see that a $1_1 2_2$ pattern will occur if there is any entry in $E_1$ that is of smaller numerical value than one in $E_2$. Thus, all entries in $E_1$ are larger than those in $E_2$. Suppose $|E_1 \cup E_2| = i$ for some $0 \le i \le k$. Then, there are $i+1$ ways to partition the entries into $E_1$ and $E_2$ (i.e., to find a $j = 0, 1 \ldots, i$ such that the smallest $j$ entries are colored 2 and the rest are colored 1).

Thus, still with $(k-1)!$ ways to wrap all entries around a circle, there are $\binom{k}{i}$ ways to choose entries of $E_1 \cup E_2$. After identifying $j$ there are $(m-2)^{k-i}$ ways to color the remaining entries. Consequently the number of such permutations is

$$\begin{aligned}
&(k-1)!\sum_{i=0}^{k}\left((m-2)^{k-i}(i+1)\binom{k}{i}\right)\\
&\quad= (k-1)!\sum_{i=1}^{k}\left((m-2)^{k-i}\,i\binom{k}{i}\right) + (k-1)!\sum_{i=0}^{k}\left((m-2)^{k-i}\binom{k}{i}\right)\\
&\quad= (k-1)!\sum_{i=1}^{k}\left((m-2)^{(k-1)-(i-1)}k\binom{k-1}{i-1}\right) + (k-1)!\sum_{i=0}^{k}\left((m-2)^{k-i}\binom{k}{k-i}\right)
\end{aligned}$$

$$= k!\,((m-2)+1)^{k-1} + (k-1)!\,((m-2)+1)^{k}$$
$$= (k-1)!\,(m-1)^{k} + k!\,(m-1)^{k-1} = (k-1)!\,(m-1)^{k-1}(k+m).$$

As a result we have the following.

**Theorem 5.2.** *The number of circular permutations in $\mathcal{S}_{k,m}$ that avoid a given nonmonochromatic length-2 pattern is*

$$(k-1)!\,(m-1)^{k-1}(k+m).$$

***Avoiding monochromatic patterns of length 3 in $\mathcal{S}_{k,m}$.*** All circular permutations of length 3 are equivalent under circular shift and reverse. So we will only consider, without loss of generality, the pattern $1_1 3_1 2_1$. It is known that in the noncolored case the only circular permutation that avoids 132 is the identity permutation.

Let $\pi$ be a permutation that avoids $1_1 3_1 2_1$ in $\mathcal{S}_{k,m}$, and let $E_1$ (of cardinality $i = 0, 1, \dots, k$) be the set of entries of color 1, then:

- If $i \geq 3$, there is only one way to order the entries in $E_1$ (i.e., in increasing order). Starting with $(k-1)!$ ways to wrap the $k$ entries (regardless of color) around a circle, only one of the $(i-1)!$ orderings of the entries in $E_1$ can be chosen. Noting that there are $\binom{k}{i}$ ways to pick the numerical values of the entries in $E_1$ and $(m-1)^{k-i}$ ways to assign colors to the remaining entries, we have the number of such permutations as
$$\sum_{i=3}^{k}\left(\binom{k}{i}\frac{(k-1)!}{(i-1)!}(m-1)^{k-i}\right).$$
- If $i \leq 2$, then there are $\binom{k}{i}$ ways to pick these $i$ entries, $(k-1)!$ ways to wrap all the entries around the circle and $(m-1)^{k-i}$ ways to color the other entries. The number of such permutations is
$$\sum_{i=0}^{2}\left(\binom{k}{i}(k-1)!(m-1)^{k-i}\right).$$

We may combine the above two formulas and conclude the following.

**Theorem 5.3.** *The number of circular permutations in $\mathcal{S}_{k,m}$ that avoid a given monochromatic length-3 pattern is*

$$(k-1)!\,(m-1)^{k} + \sum_{i=1}^{k}\left(\binom{k}{i}\frac{(k-1)!}{(i-1)!}(m-1)^{k-i}\right).$$

***Wilf classes.*** Theorems 5.1 and 5.2 imply that for any $m$-colored pattern of length 2, say $\phi$, the number of $\phi$-avoiding circular permutations in $\mathcal{S}_{k,m}$ is $(k-1)!(m-1)^{k-1}(k+m)$. This interesting (and perhaps a little surprising) observation is analogous to the findings in [Mansour 2001] in noncircular case.

This also implies that there is only one Wilf class of colored circular permutations when restricted by one pattern of length 2.

## 6. Concluding remarks and additional questions

In this short note, we considered questions related to superpatterns, pattern packing, and pattern avoidance in colored circular permutations. We presented some elementary observations, especially those generalized from previously established results on colored (but not circular) permutations, on each of these three questions. Many interesting questions remain to be further explored.

In Section 3, we introduced generalizations of a few facts on colored superpatterns. The arguments of these generalizations follow from direct adjustment of those in [Gray and Wang 2016]. There are, however, also some constructive proofs that cannot be directly generalized to circular cases. It would be interesting to further investigate them.

The several theorems in Section 4 claim that for patterns with two or three colored blocks their corresponding optimal colored circular permutations include at least one with exactly the same format (in terms of the colored blocks). With these facts, one may easily calculate the packing densities of various patterns. It is not clear whether this is true for more colored blocks. It is also mentioned that, for the pattern $\rho = 1_r 2_b$, there exist optimal permutations that have a different format. It seems likely that this is the only such case, though we do not have a proof yet.

The numbers of permutations avoiding various given patterns, as studied in Section 5, lead to an interesting statement on the Wilf classes of colored circular permutations restricted by 2-letter patterns. It appears to be much more complicated to examine the same problem for colored circular permutations restricted by longer patterns or more than one 2-letter patterns.

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