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Classifying linear operators over the octonions

Alex Putnam and Tevian Dray

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We classify linear operators over the octonions and relate them to linear equations with octonionic coefficients and octonionic variables. Along the way, we also classify linear operators over the quaternions, and show how to relate quaternionic and octonionic operators to real matrices. In each case, we construct an explicit basis of linear operators that maps to the canonical (real) matrix basis; in contrast to the complex case, these maps are surjective. Since higher-order polynomials can be reduced to compositions of linear operators, our construction implies that the ring of polynomials in one variable over the octonions is isomorphic to the product of eight copies of the ring of real polynomials in eight variables.

1. Introduction

The simplest equations are linear and homogeneous; think $y = mx$. However, even linear equations of this form become complicated over number systems other than the reals. What would happen if $mx \neq xm$, or $m(nx) \neq (mn)x$? To address such questions, we analyze here multiplicative operators like mx over the four division algebras, namely the familiar real ($\mathbb{R}$) and complex ($\mathbb{C}$) numbers, and the less familiar quaternions ($\mathbb{H}$), which are not commutative, and octonions ($\mathbb{O}$), which are neither commutative nor associative.

In the real and complex cases, it is straightforward to rewrite such operators as real matrices. As explained in Section 2, we can generate all such matrices over the reals, but not over the complexes. However, it is initially somewhat surprising to discover that in the remaining cases we can again generate all such matrices, as discussed in Sections 2 and 3. Finally, we discuss some consequences of our work in Section 4, including the immediate generalization to higher-order polynomials.

So far as we are aware, there has not been much previous investigation of octonionic polynomials, linear or otherwise. Serôdio [2007; 2010] considered polynomials with coefficients in $\mathbb{O}$, but only for real variables. Rodríguez-Ordónez [2010] classified products of linear equations over $\mathbb{O}$, and Datta and Nag [1987]

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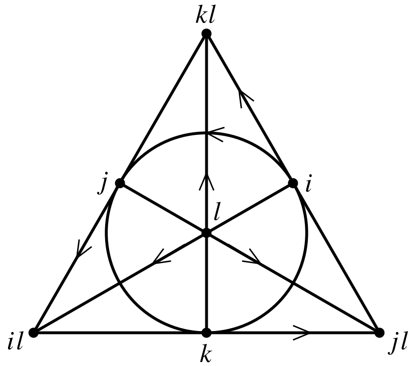

Figure 1. Octonionic multiplication in the Fano plane. Each of the seven oriented lines represents a quaternionic subalgebra; products of two elements on such a line yield $\pm$ the third element, with the sign determined by the arrows.

analyzed the topology of the roots of (some) polynomials over $\mathbb{O}$. In this work, we provide a classification of all linear equations over $\mathbb{O}$, and discuss its consequences for polynomials.

Complex numbers can be thought of as a pair of real numbers, the real and imaginary parts; thus, $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\, i$, so that $\mathbb{C} \cong \mathbb{R}^2$ as a vector space. In addition, $\mathbb{C}$ admits a product, defined by $i^2 = -1$. Similarly, the quaternions satisfy $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\, j$, with multiplication defined by

$$i^2 = j^2 = -1, \quad ji = -ij, \tag{1}$$

from which it follows by associativity that $k = ij$ also satisfies $k^2 = -1$. Multiplication of imaginary quaternions is much like the cross product, and in fact predates it historically. Finally, the octonions (see, e.g., [Dray and Manogue 2015]) satisfy $\mathbb{O} = \mathbb{H} + \mathbb{H}\,\ell$, where $\ell^2 = -1$; the complete multiplication table can be represented via the oriented Fano plane, as shown in Figure 1. It is easy to check that the octonions are not associative; for instance, $(ij)\ell = k\ell = -i(j\ell)$.

Each of the number systems $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is a *composition algebra*, admitting the operation of conjugation,

$$\bar{x} = 2\,\mathrm{Re}(x) - x \tag{2}$$

and an inner product

$$|x|^2 = x\bar{x} \tag{3}$$

satisfying

$$|xy| = |x|\,|y|. \tag{4}$$

Each $\mathbb{K}$ is also a *division algebra*, that is, a vector space on which a compatible multiplication is defined, and in which all nonzero elements are invertible. Explicitly, the multiplicative inverse of $0 \neq x \in \mathbb{K}$ is given by

$$x^{-1} = \frac{\bar{x}}{|x|^2}. \tag{5}$$

The Hurwitz theorem [1922] asserts that these four algebras are the only (positive-definite) composition algebras over the reals.

2. Real, complex and quaternionic linear operators

We now explore certain linear operators over each division algebra $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Let $\mathcal{L}(\mathbb{K})$ be the set of all multiplicative linear operators from $\mathbb{K}$ to $\mathbb{K}$, that is, all real-linear operators from $\mathbb{K}$ to $\mathbb{K}$ that can be realized using multiplication (and addition) within $\mathbb{K}$. More precisely, $\mathcal{L}(\mathbb{K})$ is the group generated by the left and right translations

$$\begin{aligned} m_L &: \mathbb{K} \to \mathbb{K}, & m_R &: \mathbb{K} \to \mathbb{K}, \\ & x \mapsto mx, & & x \mapsto xm \end{aligned} \tag{6}$$

for $m \in \mathbb{K}$. These translations are linear over $\mathbb{R}$ by distributivity and the commutativity and associativity of elements of $\mathbb{R}$ in $\mathbb{K}$. That is,

$$m_L(x + ry) = m_L(x) + r m_L(y) \tag{7}$$

for $x, y \in \mathbb{K}$ and $r \in \mathbb{R}$, and similarly for m_R. Thus, $\mathcal{L}(\mathbb{K})$ must have a matrix representation

$$\pi_{\mathbb{K}} : \mathcal{L}(\mathbb{K}) \to M_{\dim(\mathbb{K})}(\mathbb{R}), \tag{8}$$

where $M_k(\mathbb{R})$ denotes the set of $k \times k$ real matrices.

Since elements of $\mathbb{R}$ associate and commute, any linear operator over $\mathbb{R}$ can be expressed in the form

$$x \mapsto mx, \tag{9}$$

where $m, x \in \mathbb{R}$. For reasons that will become obvious as we lose commutativity and associativity, we will refer to this linear operator as "mx"; that is, we use the image of the operator acting on a "place-holder" variable, x, (also) as the name of the operator. In this sense, $mx \in \mathcal{L}(\mathbb{R})$. Since elements of $M_1(\mathbb{R})$ are matrices of the form $M = (m)$, we have the natural definition

$$\pi_{\mathbb{R}}(mx) = (m). \tag{10}$$

Thus, the set of linear operators on $\mathbb{R}$ is equivalent to the set of real 1×1 matrices, and $\pi_{\mathbb{R}}$ is the trivial map.

Complex numbers also commute and associate, so linear operators over $\mathbb{C}$ can again be expressed in the form (9), where now $m, x \in \mathbb{C}$. Separating each complex number into real and imaginary parts, e.g., $x = x_1 + x_2\,i$, and mapping $\mathbb{C}$ into $\mathbb{R}^2$ in the natural way,

$$x_1 + x_2\,i \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{11}$$

and noting that

$$(m_1 + m_2\,i)(x_1 + x_2\,i) = m_1 x_1 - m_2 x_2 + (m_1 x_2 + m_2 x_1)\,i \tag{12}$$

brings the linear operator to the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} m_1 x_1 - m_2 x_2 \\ m_1 x_2 + m_2 x_1 \end{pmatrix} \tag{13}$$

so that

$$\pi_{\mathbb{C}}(mx) = \begin{pmatrix} m_1 & -m_2 \\ m_2 & m_1 \end{pmatrix}. \tag{14}$$

Thus, our set of linear operators over $\mathbb{C}$ has only two degrees of freedom, namely the real and imaginary parts of the coefficient m. On the other hand, the set $M_2(\mathbb{R})$ is a vector space with four (real) degrees of freedom. Therefore, there are real 2×2 matrices that cannot be expressed as a (complex-)linear operator over $\mathbb{C}$. We have therefore shown that $\pi_{\mathbb{C}} : \mathcal{L}(\mathbb{C}) \to M_2(\mathbb{R})$ cannot be a surjective map. Some simple examples of real 2×2 matrices that are not in the image of $\pi_{\mathbb{C}}$ are projections and complex conjugation.

If we look to the quaternions, we finally start to find more complicated linear operators. Since the quaternions do not commute, all multiplicative linear operators over $\mathbb{H}$ are sums of terms of the form

$$x \mapsto pxq, \tag{15}$$

where $p, q \in \mathbb{H}$. Since we can expand each quaternion p, q, in terms of a basis $\{1, i, j, k\}$ and then distribute over the expanded coefficients, we see that every linear operator over $\mathbb{H}$ can be expressed as a linear combination of terms of the form

$$x \mapsto e_m x e_n \tag{16}$$

for distinct combinations $e_m, e_n \in \{1, i, j, k\}$. Therefore, we only need to consider coefficients that are basis elements of $\mathbb{H}$. Expanding each quaternion with respect to our basis, e.g., $x = x_1 + x_2\,i + x_3\,j + x_4\,k$, and mapping $\mathbb{H}$ into $\mathbb{R}^4$ by analogy with (11) leads immediately to, for instance,

$$\pi_{\mathbb{H}}(px) = \begin{pmatrix} p_1 & -p_2 & -p_3 & -p_4 \\ p_2 & p_1 & -p_4 & p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{pmatrix}. \tag{17}$$

Although this particular operator has only four (real) degrees of freedom, it is now easy to verify [Putnam 2017] that $\pi_{\mathbb{H}}$ maps the set of all 16 operators $\{e_m x e_n\}$ to a basis of $M_4(\mathbb{R})$.[1] Thus, $\pi_{\mathbb{H}}$ must be a bijection between the linear operators over $\mathbb{H}$ and $M_4(\mathbb{R})$.

An explicit pairing of each elementary matrix in $M_4(\mathbb{R})$ with a corresponding multiplicative linear operator over $\mathbb{H}$ is given in [Putnam 2017]. Intriguing examples are

$$x - ixi - jxj - kxk = 4\,\mathrm{Re}(x), \tag{18}$$

$$x + ixi + jxj + kxk = -2\bar{x}, \tag{19}$$

each of which can be verified (or discovered!) by applying $\pi_{\mathbb{H}}$. Conjugation is a linear map over $\mathbb{H}$!

3. Octonionic linear operators

We are now ready to look at $\mathcal{L}(\mathbb{O})$, the multiplicative linear operators over $\mathbb{O}$. If we consider operators of the form (15) with $p, q, x \in \mathbb{O}$, then, because $\mathbb{O}$ is not associative, we are really considering two different operators, one of the form $x \mapsto (px)q$, and the other of the form $x \mapsto p(xq)$, unless p, q are in a complex subalgebra of $\mathbb{O}$ (since the octonions are alternative). We can, however, continue to nest more coefficients outside of these two terms. Just as before, because we can distribute over the expanded form of $x \in \mathbb{O}$, we only need to consider linear operators with basis elements as coefficients. Mapping $\mathbb{O}$ into $\mathbb{R}^8$ again gives a natural definition of, for example,

$$\pi_{\mathbb{O}}(px) = \begin{pmatrix} p_1 & -p_2 & -p_3 & -p_4 & -p_5 & -p_6 & -p_7 & -p_8 \\ p_2 & p_1 & -p_4 & p_3 & -p_6 & p_5 & p_8 & -p_7 \\ p_3 & p_4 & p_1 & -p_2 & p_7 & p_8 & -p_5 & -p_6 \\ p_4 & -p_3 & p_2 & p_1 & p_8 & -p_7 & p_6 & -p_5 \\ p_5 & p_6 & -p_7 & -p_8 & p_1 & -p_2 & p_3 & p_4 \\ p_6 & -p_5 & -p_8 & p_7 & p_2 & p_1 & -p_4 & p_3 \\ p_7 & -p_8 & p_5 & -p_6 & -p_3 & p_4 & p_1 & p_2 \\ p_8 & p_7 & p_6 & p_5 & -p_4 & -p_3 & -p_2 & p_1 \end{pmatrix}. \tag{20}$$

Because we can nest the coefficients of x, we need to count how many nestings we are likely to need to show whether $\pi_{\mathbb{O}}$ is surjective. If we consider operators of

[1] Alternatively, one can verify by direct computation that the matrix $\pi_{\mathbb{H}}\left(\sum a_{m,n} e_m x e_n\right)$ is

$$\begin{pmatrix} a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4} & -a_{1,2}-a_{2,1}+a_{3,4}-a_{4,3} & -a_{1,3}-a_{2,4}-a_{3,1}+a_{4,2} & -a_{1,4}+a_{2,3}-a_{3,2}-a_{4,1} \\ a_{1,2}+a_{2,1}+a_{3,4}-a_{4,3} & a_{1,1}-a_{2,2}+a_{3,3}+a_{4,4} & a_{1,4}-a_{2,3}-a_{3,2}-a_{4,1} & -a_{1,3}-a_{2,4}+a_{3,1}-a_{4,2} \\ a_{1,3}-a_{2,4}+a_{3,1}+a_{4,2} & -a_{1,4}-a_{2,3}-a_{3,2}+a_{4,1} & a_{1,1}+a_{2,2}-a_{3,3}+a_{4,4} & a_{1,2}-a_{2,1}-a_{3,4}-a_{4,3} \\ a_{1,4}+a_{2,3}-a_{3,2}+a_{4,1} & a_{1,3}-a_{2,4}-a_{3,1}-a_{4,2} & -a_{1,2}+a_{2,1}-a_{3,4}-a_{4,3} & a_{1,1}+a_{2,2}+a_{3,3}-a_{4,4} \end{pmatrix}$$

and then check that the 16 degrees of freedom (the matrix coefficients of $a_{m,n}$) are independent.

the form $x \mapsto pxq$ then at first sight we have $8^2 = 64$ such operators. However, the lack of associativity means that there are $2\binom{7}{2} = 42$ cases where we must count both possible orders of multiplication, resulting in $64 + 42 = 106$ operators with distinct orderings of coefficients. Since $\dim M_8(\mathbb{R}) = 64$, these 106 operators cannot be linearly independent, but it is not obvious whether they span $\mathcal{L}(\mathbb{O})$. Since right multiplication can be expressed in terms of (nested) left multiplication [Conway and Smith 2003], we will instead consider operators with coefficients only on the left. We have the identity operator, $x \mapsto x$, and seven operators of the form $x \mapsto e_n x$ with $e_n \in \{i, j, k, i\ell, j\ell, k\ell, \ell\}$. If we consider one nested coefficient, then we have the form $x \mapsto e_n(e_m x)$, again with $e_n \neq 1 \neq e_m$ and $\binom{7}{2} = 21$ new operators. These singly nested products were shown in [Manogue and Schray 1993] to generate the orthogonal group SO(7).

Next, we consider two nestings, which yields $\binom{7}{3} = 35$ more operators. Amazingly, this process gives us a total of $1 + 7 + 21 + 35 = 64$ distinct (representations of) operators in $\mathcal{L}(\mathbb{O})$! It was shown in [Putnam 2017] that these 64 linear operators are in fact linearly independent; an explicit pairing with the canonical basis of $M_8(\mathbb{R})$ was also given. Thus, $\pi_{\mathbb{O}}$ is surjective, and doubly nested representations are precisely enough to express all elements $\mathcal{L}(\mathbb{O})$.

In the previous cases, we were only able to construct linear operators for $\dim(\mathbb{K})^2$ different combinations of coefficients of basis elements, because each underlying space was associative. In $\mathbb{O}$, we can construct the same linear operators with different combinations of coefficients of basis elements. So, operators that appear to be different may have the same image $\pi_{\mathbb{O}}$, and thus in fact correspond to different representations of the same element of $\mathcal{L}(\mathbb{K})$.[2] It is now straightforward to show that $\mathcal{L}(\mathbb{O})$ forms a group under operator composition, and that the map $\pi_{\mathbb{O}} : \mathcal{L}(\mathbb{O}) \to M_8(\mathbb{R})$ is a bijection. In particular, it then follows that right multiplication can be expressed in terms of left multiplication, thus verifying the result of [Conway and Smith 2003], and this can be done explicitly by finding a linear combination of the basis given in [Putnam 2017] that yields the same matrix.

Since $\pi_{\mathbb{O}}$ is a surjective map, there must exist elements $f_{n,m} \in \mathcal{L}(\mathbb{O})$ such that $f_{n,m}(x) = x_n e_m$ for $1 \leq n \leq 8$ and $e_m \in \{1, i, j, k, i\ell, j\ell, k\ell, \ell\}$. Some other intriguing elements of $\mathcal{L}(\mathbb{O})$ are given by $x - i(j(kx))$, which projects out the quaternionic part of x, and $x - ixi$, which projects out the complex part of x. It is a useful exercise to work out a representation of the latter operator in terms of nested left multiplication! Again, these assertions can be verified or discovered by applying $\pi_{\mathbb{O}}$.

[2]An alternative treatment, as in [Putnam 2017], would regard $\mathcal{L}(\mathbb{K})$ as being freely generated by left and right translations, then define an equivalence relation $L \sim M$ on elements $L, M \in \mathcal{L}(\mathbb{K})$ if $\pi_{\mathbb{O}}(L) = \pi_{\mathbb{O}}(M)$. The relation $\sim$ is clearly an equivalence relation, since it is defined by equality of matrices, and what we here call $\mathcal{L}(\mathbb{O})$ would instead be the quotient $\mathcal{L}(\mathbb{O})/\sim$.

4. Conclusion

We have shown that the lack of commutativity of the quaternions, and the lack of associativity of the octonions, conspire to provide just enough degrees of freedom that multiplicative linear operators do indeed generate all real-linear maps in those cases — despite the fact that they do not do so in the complex case. In the quaternionic case, the extra degrees of freedom manifest themselves when considering two-sided operators, whereas in the octonionic case it is the nested nature of iterated multiplication that generates the necessary degrees of freedom. Along the way, we have verified the assertion stated without proof in Section 3 that right multiplication can be expressed in terms of nested left multiplication.

In the octonionic case, we have further shown that it takes precisely three iterated products to generate all 64 independent real-linear maps, noting that $\binom{7}{0}+\binom{7}{1}+\binom{7}{2}+\binom{7}{3}=64$. This result has an intriguing application to the Clifford algebra Cl(6), which can be represented precisely as the 64-dimensional matrix algebra $M_8(\mathbb{R})$. As has been noted by Furey [2014], it is therefore possible to represent Cl(6) entirely in terms of octonionic multiplication, with possible applications to particle physics; see, e.g., [Dray and Manogue 2015].

Having classified multiplicative linear operators over $\mathbb{O}$, we could consider higher-degree terms, that is, octonionic polynomials. By the distributive law, and because real numbers commute and associate with octonions, we can expand each such term (both coefficients and variables) with respect to a basis. Just as there are $8=\binom{8}{1}$ (real-)independent components of x, and hence $8\times 8=64$ independent linear operators on $\mathbb{O}$, there are similarly $\binom{8}{2}+8=36$ quadratic "components" of x^2, where the last "8" counts coefficients that are squared. Thus, the most general quadratic operator maps x to a linear combination of the $8\times 36=288$ terms $x_m x_n e_p$, where $1\le m\le n\le 8$ and $1\le p\le 8$. Furthermore, we can realize each such operator (in multiple ways) as a composition of the linear operators $f_{m,n}$, and hence in terms of octonionic multiplication. A similar process can be applied to higher-order terms. It is obvious that any polynomial over $\mathbb{O}$ can be reinterpreted as eight real polynomials in eight variables; our construction shows that the converse is also true, so that $\mathbb{O}[x]\cong(\mathbb{R}[x_1,\ldots,x_8])^8$.

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