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The k-diameter component edge connectivity parameter

Nathan Shank and Adam Buzzard

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We focus on a network reliability measure based on edge failures and considering a network operational if there exists a component with diameter k or larger. The *k-diameter component edge connectivity parameter* of a graph is the minimum number of edge failures needed so that no component has diameter k or larger. This implies each resulting vertex must not have a k-neighbor. We give results for specific graph classes including path graphs, complete graphs, complete bipartite graphs, and a surprising result for perfect r-ary trees.

1. Introduction

Network reliability and graph connectivity parameters have been studied for many years. The network reliability measure can vary greatly based on the type of application being considered. In particular networks, the vulnerabilities of particular pieces of the network often influence the parameter used to measure reliability. In particular cases, nodes or vertices may fail or become inoperable; in other cases, the edges or connections between vertices may fail or become inoperable and in some cases both the nodes and the edges may fail. See [Boesch et al. 2009] for a survey of recent results and techniques.

In general, network reliability measures are driven by two different yet connected concepts. First, we need to know what objects are prone to failure: edges, vertices, or both. Second, we need to know what the requirements are to make a network functional. Stated differently, we need to know what objects fail and what characterizes a failure state for a network.

Vertex connectivity and *edge connectivity* are two of the original network reliability measures which have been studied extensively. The vertex connectivity parameter is the minimum number of vertices that must be deleted so that the resulting graph is disconnected. Similarly the edge connectivity parameter measures

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the minimum number of edges that must be deleted so that the resulting graph is disconnected. These parameters have been generalized to other reliability measures based on different characterizations of failure states for networks. For example, the *component order vertex connectivity parameter* is the minimum number of vertices that must be deleted so that the resulting graph has all components of order less than some value k (see [Boesch et al. 1998; 1999] for example). Similarly the *component order edge connectivity parameter* is the minimum number of edges that must be deleted so that the resulting graph has all components of order less than some value k (see [Boesch et al. 2006; 2007] for example).

Conditional connectivity was studied by Frank Harary [1983]. It requires each component of a disconnected graph to have a chosen property P. Thus if P is any property of a graph $G = (V, E)$ and $S \subset V(G)$, then the P-connectivity of G is the minimum $|S|$ such that $G - S$ is disconnected and every component of $G - S$ has property P. Similarly we can define the edge conditional connectivity parameter of G if we consider edge deletions rather than vertex deletions.

In this paper, we focus our attention on edge failures and consider a graph to be in a failure state if no vertex has a neighbor of a fixed distance. In other words, we study the minimum number of edges that can fail in order to produce a graph which has all components with a diameter less than some fixed value. In this particular case a network would be operational if there exists a component with a sufficiently large diameter.

One important application of such a parameter centers around the spread of disease or genetic traits. If a particular disease or genetic trait only becomes active after k successive transmissions, then we would want to stop the spread so that components in the network (tree) have diameter less than k. This will be explored more in Section 3B.

2. Background and definitions

Throughout this paper, let $G = (V, E)$ be a simple graph with vertex set V and edge set E. For any set A, let $|A|$ denote the cardinality of A. If $D \subset E$, let $G - D$ denote the subgraph of G containing the vertex set V and the edge set $E - D$. Thus $G - D = (V, E - D)$.

Throughout the paper, unless otherwise specified, we will assume that n, r, l, and k are all positive integers. We will also use the conventions of notation adapted from [West 1996]. A pair of vertices u, v are said to be k-neighbors if the distance between u and v is k, written as $d(u, v) = k$.

Definition 2.1. Let $G = (V, E)$ be a graph and k be a positive integer. A set $D \subseteq E$ is a *k-diameter component edge disconnecting set* if $G - D$ has all components of diameter less than k.

This means that an edge set D is a *k-diameter component edge disconnecting set* if no vertex in $G - D$ has a k-neighbor. If D is a *k-diameter component edge disconnecting set* then $G - D$ is said to be a *failure state.*

Definition 2.2. Given a graph $G = (V, E)$ and a positive integer k, the *k-diameter component edge connectivity parameter* of G, denoted by $\text{CE}_k(G)$, is the size of the smallest k-diameter component edge disconnecting set.

Thus, the k-diameter component edge connectivity parameter is the size of the smallest edge set D such that $G - D$ is a failure state.

3. Results

When $k = 1$, a failure state will occur if no vertex has a 1-neighbor. In order for this to occur every edge must be removed. Thus $\text{CE}_1(G) = |E|$ for every graph $G = (V, E)$. Therefore for the remainder of the paper we will assume that $k \geq 2$.

In Section 3A we will show some easy results for some simple graph classes, particularly path graphs, complete graphs, and complete bipartite graphs. In Section 3B1 we will consider perfect r-ary trees.

3A. *Simple graphs.*

3A1. *Path graphs.* The first type of graph we will consider is a path on n vertices, denoted by P_n. We can label the edges consecutively from 1 to $n-1$ starting at a pendant edge. For a component to have a diameter less than k, it can have at most $k-1$ edges. If we delete every edge whose label is a multiple of k, then the remaining components all have $k-1$ edges, except for possibly one component which could have less than $k-1$ edges. Therefore the diameter of each component will be less than k. Hence we see $\text{CE}_k(P_n) \leq \lfloor (n-1)/k \rfloor$.

Since path graphs are trees, every edge deletion creates one new component. Since we cannot have components of length k in a failure state, we need at least one edge deletion in every k-edge disjoint connected subpath. Hence $\text{CE}_k(P_n) \geq \lfloor (n-1)/k \rfloor$. These two observations imply the following:

Theorem 3.1. *For every positive integer n,*

$$\text{CE}_k(P_n) = \left\lfloor \frac{n-1}{k} \right\rfloor.$$

3A2. *Complete graphs.* Since the diameter of K_n is 1, K_n is already a failure state. Thus we see the following obvious result:

Theorem 3.2. *For every positive integer n,*

$$\text{CE}_k(K_n) = 0.$$

3A3. *Complete bipartite Graphs.* Consider the complete bipartite graph $K_{a,b} = (V, E)$ with parts A and B, where $V = A \cup B,\ A \cap B = \varnothing,\ |A| = a > 0$ and $|B| = b > 0$. Recall that the diameter of a complete bipartite graph is 2 unless $a = b = 1$, in which case the diameter is 1. If $k > 2$, then $K_{a,b}$ is already a failure state. If $k = 2$, then the size of the largest subgraph in a failure state is the size of the maximum matching in $K_{a,b}$, which is $\min\{a, b\}$. So the number of edges that must be deleted to produce a failure state is $\min\{a, b\}$ less than the total number of edges. Therefore we have the following theorem:

Theorem 3.3. *For every pair of positive integers $a \le b$,*

$$\mathrm{CE}_k(K_{a,b}) = \begin{cases} 0 & \text{if } k > 2, \\ a(b-1) & \text{if } k = 2. \end{cases}$$

3B. ***Trees.***

3B1. *Perfect r-ary trees.* We will now consider perfect r-ary trees.

Definition 3.4. Let $T_{r,l} = (V, E)$ denote a perfect r-ary tree with height l, where

$$V = \{v_{i,j} : 1 \le i \le l+1,\ 1 \le j \le r^{(l+1)-i}\}, \quad \text{and}$$
$$E = \{(v_{i,j}, v_{i-1,m}) : 2 \le i \le l+1,\ 1 \le j \le r^{(l+1)-i},\ (j-1)r + 1 \le m \le jr\}.$$

We will say that vertex $v_{i,h} \in V(T_{r,l})$ is on *level i*. Notice we are using the unconventional notation that the root vertex of the full complete tree is on level $l+1$ and the leaves are on level 1.

In order to separate the tree into failure states we need to know the distance between vertices. The following lemma shows a lower bound for the distance between two vertices in the same level.

Lemma 3.5. *Assume $T_{r,l} = (V, E)$ and $v_{i,j}, v_{i,j+pr^{n-1}} \in V$ for some positive integers i, j, n, and p. Then*

$$d(v_{i,j}, v_{i,j+pr^{n-1}}) \ge 2n.$$

Proof. We will proceed by induction on n. Consider the case when $n = 1$.

Since $v_{i,j}$ and $v_{i,j+p}$ are both on level i, they are not adjacent. Since any two vertices of a tree are connected by a path, we conclude $d(v_{i,j}, v_{i,j+p}) \ge 2$.

Assume there exists a positive integer n such that for any pair $v_{a,b}, v_{a,b+pr^{n-1}} \in V$,

$$d(v_{a,b}, v_{a,b+pr^{n-1}}) \ge 2n.$$

Consider a pair of vertices, $v_{i,j},\ v_{i,j+qr^n} \in V$ for some positive integer q. The unique path from $v_{i,j}$ to $v_{i,j+qr^n}$ must contain vertices

$$v_{i+1,\lceil j/r \rceil} \quad \text{and} \quad v_{i+1,\lceil j/r \rceil + qr^{n-1}}.$$

By induction we know

$$d(v_{i+1,\lceil j/r\rceil}, v_{i+1,\lceil j/r\rceil+qr^{n-1}}) \geq 2n.$$

Therefore by the uniqueness of paths in trees we see

$$d(v_{i,j}, v_{i,j+qr^n}) = d(v_{i+1,\lceil j/r\rceil}, v_{i+1,\lceil j/r\rceil+qr^{n-1}}) + 2 \geq 2(n+1). \qquad \square$$

To find $\mathrm{CE}_k(T_{r,l})$ we will find a set of vertices $V' \subset V$ such that the distance between any two vertices in V' is at least k, therefore finding a lower bound $|V'|$ for the number of components in a failure state for $T_{r,l}$. We will then show that you can make $T_{r,l}$ a failure state by removing $|V'|$ edges.

The following lemma produces a set V' of vertices such that the distance between any two vertices in V' is at least k.

Lemma 3.6. *Let $k \in \mathbb{Z}^+$. Suppose $T_{r,l} = (V, E)$ and $V' \subseteq V$ such that*

$$V' = \left\{ v_{yk+1,1+zr^{\lfloor (k-1)/2\rfloor}} : 0 \leq y \leq \left\lfloor \frac{l}{k} \right\rfloor,\ 0 \leq z \leq \left\lceil \frac{r^{l-yk}}{r^{\lfloor (k-1)/2\rfloor}} \right\rceil - 1 \right\}.$$

Then for all distinct $u, v \in V'$,

$$d(u, v) \geq k.$$

Proof. Assume $u, v \in V'$. Consider the following two cases:

Case 1: Assume u and v are distinct vertices in the same level of $T_{r,l}$. Thus there exist some integers i, a, and b such that $u = u_{i,1+ar^{\lfloor (k-1)/2\rfloor}}$ and $v = v_{i,1+(a+b)r^{\lfloor (k-1)/2\rfloor}}$. Then, by Lemma 3.5,

$$\begin{aligned} d(u, v) &= d(u_{i,1+ar^{\lfloor (k-1)/2\rfloor}}, v_{i,1+(a+b)r^{\lfloor (k-1)/2\rfloor}}) \\ &= d(u_{i,1+ar^{\lfloor (k-1)/2\rfloor}}, v_{i,1+ar^{\lfloor (k-1)/2\rfloor}+br^{(\lfloor (k-1)/2\rfloor+1)-1}}) \\ &\geq 2\left(\left\lfloor \tfrac{1}{2}(k-1)\right\rfloor + 1\right) \geq k. \end{aligned}$$

Case 2: Assume $u = u_{i,j}$ and $v = v_{i',j'}$ for some $i \neq i'$. Since $u, v \in V'$, we know $|i - i'| \geq k$. Therefore $d(u, v) \geq k$. $\square$

Now that we know the distance between any two vertices in V' is at least k, we need to find $|V'|$.

Lemma 3.7. *Suppose $T_{r,l} = (V, E)$ and $V' \subseteq V$ such that*

$$V' = \left\{ v_{yk+1,1+zr^{\lfloor (k-1)/2\rfloor}} : 0 \leq y \leq \left\lfloor \frac{l}{k} \right\rfloor,\ 0 \leq z \leq \left\lceil \frac{r^{l-yk}}{r^{\lfloor (k-1)/2\rfloor}} \right\rceil - 1 \right\}.$$

Then,

$$|V'| = \begin{cases} 1, & l \leq \left\lfloor \frac{1}{2}(k-1) \right\rfloor, \\ \dfrac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \dfrac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})} + 1, & nk \leq l \leq nk + \left\lfloor \frac{1}{2}(k-1) \right\rfloor, \\ \dfrac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \dfrac{1-(r^{-k})^{\lfloor l/k \rfloor + 1}}{1-(r^{-k})}, & \textit{else}, \end{cases}$$

where n is a positive integer.

Proof. Summing over all possible choices for y and z we see

$$|V'| = \sum_{y=0}^{\lfloor l/k \rfloor} \sum_{z=0}^{\lceil R_y \rceil - 1} 1,$$

where $R_y = r^{l-yk}/r^{\lfloor (k-1)/2 \rfloor}$. Consider the following three cases:

Case 1: If $l \leq \left\lfloor \frac{1}{2}(k-1) \right\rfloor$, then $\lfloor l/k \rfloor = 0$ which implies y can only be zero. Thus

$$\lceil R_y \rceil - 1 = \lceil R_0 \rceil - 1 = 0.$$

Therefore

$$|V'| = \sum_{y=0}^{0} \sum_{z=0}^{0} 1 = 1.$$

Case 2: Assume there exists a positive integer n such that $nk \leq l \leq nk + \left\lfloor \frac{1}{2}(k-1) \right\rfloor$.

If $y = n$, then $\lceil R_y \rceil = \lceil R_n \rceil = 1$ since $0 \leq l - nk \leq \left\lfloor \frac{1}{2}(k-1) \right\rfloor$.

If $y < n$, then $y + 1 \leq n$, which implies $k(y+1) \leq kn \leq l$. Therefore $k \leq l - yk$, which implies

$$\lceil R_y \rceil = R_y.$$

Since $\lfloor l/k \rfloor = n$,

$$|V'| = \sum_{y=0}^{n} \sum_{z=0}^{\lceil R_y \rceil - 1} 1 = \sum_{y=0}^{n-1} \sum_{z=0}^{R_y - 1} 1 + \sum_{z=0}^{\lceil R_n \rceil - 1} 1$$
$$= \left(\sum_{y=0}^{n-1} R_y \right) + 1 = \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})} + 1.$$

Case 3: Assume $nk + \left\lfloor \frac{1}{2}(k-1) \right\rfloor + 1 \leq l \leq (n+1)k - 1$ for some nonnegative integer n.

Note that $\left\lfloor \frac{1}{2}(k-1) \right\rfloor \leq l - nk$. Then for all $y \leq n$,

$$\lceil R_y \rceil = R_y.$$

Since $\lfloor l/k \rfloor = n$,

$$|V'| = \sum_{y=0}^{n} \sum_{z=0}^{\lceil R_y \rceil - 1} 1 = \sum_{y=0}^{n} \sum_{z=0}^{R_y - 1} 1$$
$$= \sum_{y=0}^{n} R_y = \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1 - (r^{-k})^{\lfloor l/k \rfloor + 1}}{1 - (r^{-k})}. \qquad \square$$

We have now constructed a set of vertices which must be in separate components in order for $T_{r,l}$ to be a failure state (Lemma 3.6) and calculated the size of this vertex set (Lemma 3.7). We will now construct a set of edges that, when deleted, ensure these vertices are in different components. The idea is not to create perfect r-ary subtrees as we might expect. Instead we allow a perfect r-ary subtree but allow its root vertex to have a path up $T_{r,l}$ until the maximum diameter allowed is achieved. This propagates up the tree so that we do not have to remove entire rows of edges very often. This "saves" edges from being deleted by creating failure components which are larger than a perfect r-ary tree of diameter $k-1$.

Lemma 3.8. *Fix r, l, and k and suppose $T_{r,l} = (V, E)$. For each integer $0 \le m \le \lceil l/k \rceil - 1$ define the sets*

$$A_m = \bigl\{(v_{i,j}, v_{i+1,\lceil j/r \rceil}) \in E : mk + \bigl\lfloor \tfrac{1}{2}(k-1) \bigr\rfloor + 1 \le i \le (m+1)k - 1, \quad j \not\equiv 1 \bmod r \bigr\},$$

and

$$B_m = \bigl\{(v_{(m+1)k,j}, v_{(m+1)k+1,\lceil j/r \rceil}) \in E : 1 \le j \le r^{l+1-(m+1)k} \bigr\}.$$

Then

$$|A_m| = r^{l+1}(r^{-mk - \lfloor (k-1)/2 \rfloor - 1} - r^{-(m+1)k}) \quad \textit{and} \quad |B_m| = r^{l+1-(m+1)k}.$$

Proof. Fix $0 \le m \le \lceil l/k \rceil - 1$. First notice the number of edges of the form $(v_{i,a}, v_{i+1,\lceil a/r \rceil})$ is the number of vertices in level i, which is r^{l+1-i}.

Now consider A_m. The total number of edges of the form $(v_{i,a}, v_{i+1,\lceil a/r \rceil})$ is r^{l+1-i}, and of these, $r^{l+1-(i+1)}$ are of the form $(v_{i,j}, v_{i+1,\lceil j/r \rceil})$, where $j \equiv 1 \bmod r$. Thus

$$|A_m| = \sum_{i=mk+\lfloor (k-1)/2 \rfloor + 1}^{(m+1)k-1} r^{l+1-i} - r^{l+1-(i+1)}$$
$$= r^{l+1}(r^{-mk-\lfloor (k-1)/2 \rfloor - 1} - r^{-(m+1)k}).$$

Next consider B_m. The set B_m contains all edges of the form $(v_{(m+1)k,j}, v_{(m+1)k+1,\lceil j/r \rceil})$. Thus

$$|B_m| = r^{l+1-(m+1)k}. \qquad \square$$

Now we are ready to use A_m and B_m to find $\mathrm{CE}_k(T_{r,l})$.

Theorem 3.9. *If r, l, and k are positive integers, then*

$$\mathrm{CE}_k(T_{r,l}) = \begin{cases} 0, & l \le \left\lfloor \frac{1}{2}(k-1) \right\rfloor, \\ \dfrac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \dfrac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})}, & nk \le l \le nk + \left\lfloor \frac{1}{2}(k-1) \right\rfloor, \\ \dfrac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \dfrac{1-(r^{-k})^{\lfloor l/k \rfloor + 1}}{1-(r^{-k})} - 1, & \text{else}, \end{cases}$$

where n is a positive integer.

Proof. Fix r, l, and k. Let $T_{r,l} = (V, E)$. There are three cases to consider:

Case 1: Assume $l \le \left\lfloor \frac{1}{2}(k-1) \right\rfloor$.

Notice that the diameter of $T_{r,l}$ is $2l$. If $l \le \left\lfloor \frac{1}{2}(k-1) \right\rfloor$, then $2l \le 2\left\lfloor \frac{1}{2}(k-1) \right\rfloor < k$, and therefore $T_{r,l}$ is already a failure state. Hence, $\mathrm{CE}_k(T_{r,l}) = 0$.

For the following two cases, consider $V' \subseteq V$ as defined in Lemma 3.6. As shown in Lemma 3.6, $d(u, v) \ge k$ for all $u, v \in V'$. Therefore, to produce a failure state, no two vertices in V' can be in the same component. Since every edge cut in a tree produces one new component, there must be at least $|V'| - 1$ edge cuts to ensure no two vertices in V' are connected. Hence $\mathrm{CE}_k(T_{r,l}) \ge |V'| - 1$.

Case 2: Assume $nk \le l \le nk + \left\lfloor \frac{1}{2}(k-1) \right\rfloor$ for some positive integer n.

By Lemma 3.7,

$$|V'| - 1 = \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})}.$$

Hence,

$$\mathrm{CE}_k(T_{r,l}) \ge \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})}.$$

For each integer $0 \le m \le \lfloor l/k \rfloor - 1$, define A_m and B_m as in Lemma 3.8.

Let $E' = \bigcup_{m=0}^{\lfloor l/k \rfloor - 1} (A_m \cup B_m)$. We will show that $G - E'$ is a failure state. Assume by way of contradiction that $G - E'$ is not a failure state. Thus there exists a path of length k in $G - E'$.

Case 2a: Assume there exists a path in $G - E'$ from a vertex in level i to a vertex in level $i + k$. Let $P = v_{i,j_0}, v_{i+1,j_1}, v_{i+2,j_2}, \dots, 1, v_{i+k,j_k}$ be such a path of length k in $G - E'$, where $j_z = \lceil j_{z-1}/r \rceil$ and $(m-1)k < i \le mk$ for some $2 \le m \le n-1$. Then, $mk < i + k \le (m+1)k$ and $i \le mk < i + k$.

Since $i \le mk < i + k$, there exist a vertex of the form $v_{mk,j_{mk-i}} \in P$ and a vertex of the form $v_{mk+1,j_{mk-i+1}} \in P$ which are adjacent. However, $(v_{mk,j_{mk-i}},$

$v_{mk+1,j_{mk-i+1}}) \in B_{m-1}$. Consequently, $(v_{mk,j_{mk-i}}, v_{mk+1,j_{mk-i+1}}) \notin G - E'$, so P is not a path in $G - E'$.

Case 2*b*: Let $P = v_{i_0,j_0}, v_{i_1,j_1}, \ldots, v_{i_k,j_k}$ be the path of length k in $G - E'$. By the definition of B_m, we know there do not exist any edges in $G - E'$ joining $v_{(m-1)k,j}$ and $v_{(m-1)k+1,\lceil j/r \rceil}$ for any integer m where $2 \le m \le n$. Therefore we can assume there exists an integer $2 \le m \le n$ such that for all $0 \le p \le k$, we have $(m-1)k + 1 \le i_p \le mk$. In other words, all the vertices of path P fall between level $(m-1)k + 1$ and level mk inclusively.

Since there are only k distinct levels between level $(m-1)k + 1$ and level mk and P has $k + 1$ vertices, this implies there exists a subpath of P of the form $v_{a,b}, v_{a+1,c}, v_{a,b'}$, where $(c-1)r + 1 \le b,\ b' \le cr$, and $b \ne b'$. Since P is of length k, we can assume without loss of generality that $d(v_{i_0,j_0}, v_{a+1,c}) \ge \lfloor \frac{1}{2}(k-1) \rfloor + 1$. This implies that $i_0 + \lfloor \frac{1}{2}(k-1) \rfloor + 1 \le a + 1$.

Since $(m-1)k + 1 \le i_0 \le mk$, we see

$$(m-1)k + \left\lfloor \tfrac{1}{2}(k-1) \right\rfloor + 1 + 1 \le i_0 + \left\lfloor \tfrac{1}{2}(k-1) \right\rfloor + 1 \le a + 1,$$

which implies

$$(m-1)k + \left\lfloor \tfrac{1}{2}(k-1) \right\rfloor + 1 \le a.$$

Also, since $a \le mk - 1$, we can see

$$(m-1)k + \left\lfloor \tfrac{1}{2}(k-1) \right\rfloor + 1 \le a \le mk - 1.$$

Since $(c-1)r+1 \le b,\ b' \le cr$ and $b \ne b'$, we know $b \not\equiv 1 \mod r$ or $b' \not\equiv 1 \mod r$. Consequently, $(v_{a,b}, v_{a+1,c}) \in A_{m-1}$ or $(v_{a,b'}, v_{a+1,c}) \in A_{m-1}$, or both are in A_{m-1}. In either case, path P is not a path in $G - E'$ since it contains an edge in A_{m-1}. Hence, $G - E'$ is a failure state.

By Lemma 3.8,

$$\begin{aligned} |E'| &= \sum_{m=0}^{n-1} (|A_m| + |B_m|) \\ &= \sum_{m=0}^{n-1} (r^{l+1}(r^{-mk-\lfloor (k-1)/2 \rfloor - 1} - r^{-(m+1)k}) + r^{l+1-(m+1)k}) \\ &= \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1 - (r^{-k})^{\lfloor l/k \rfloor}}{1 - (r^{-k})}. \end{aligned}$$

Therefore, since $G - E'$ is a failure state, we see

$$\mathrm{CE}_k(T_{r,l}) \le \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1 - (r^{-k})^{\lfloor l/k \rfloor}}{1 - (r^{-k})}.$$

Since

$$\mathrm{CE}_k(T_{r,l}) \geq |V'|-1 = \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})},$$

we see

$$\mathrm{CE}_k(T_{r,l}) = \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})}.$$

Case 3: Assume $nk + \left\lfloor \frac{1}{2}(k-1) \right\rfloor + 1 \leq l \leq (n+1)k-1$ for some positive integer n.

By Lemma 3.7,

$$|V'|-1 = \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor+1}}{1-(r^{-k})} - 1.$$

Let $A_{n^*} = \left\{ (v_{i,j}, v_{i+1,\lceil j/r \rceil}) : nk + \left\lfloor \frac{1}{2}(k-1) \right\rfloor + 1 \leq i \leq l,\ j \not\equiv 1 \bmod r \right\}$. Then,

$$|A_{n^*}| = \sum_{p=nk+\lfloor (k-1)/2 \rfloor+1}^{l} r^{l+1-p} - r^{l+1-(p+1)} = r^{l+1}(r^{-nk-\lfloor (k-1)/2 \rfloor-1} - r^{-l-1}).$$

Let $E' = \bigcup_{m=0}^{\lfloor l/k \rfloor-1} (A_m \cup B_m) \cup A_{n^*}$. We will show that $T_{r,l} - E'$ is a failure state.

First, notice $G \subset T_{r,(n+1)k}$. Let $T_{r,(n+1)k} = (V^*, E^*)$. Let $E'' \subseteq E^*$ such that

$$E'' = \bigcup_{m=0}^{\lfloor (n+1)k/k \rfloor-1} (A_m \cup B_m) = \bigcup_{0}^{\lfloor l/k \rfloor-1} (A_m \cup B_m) \cup A_n \cup B_n.$$

As shown above in Case 2, $T_{r,(n+1)k} - E''$ is a failure state.

Note that

$$A_n = \left\{ (v_{i,j}, v_{i+1,\lceil j/r \rceil}) : nk + \left\lfloor \tfrac{1}{2}(k-1) \right\rfloor + 1 \leq i \leq (n+1)k-1,\ j \not\equiv 1 \bmod r \right\}.$$

Then, since $l \leq (n+1)k-1$, we know $A_{n^*} \subseteq A_n$. Hence $E' \subseteq E''$ and $T_{r,l} - E' \subseteq T_{r,(n+1)k} - E''$. If there exists a path of length k in $T_{r,l} - E'$, then there must also exist a path of length k in $T_{r,(n+1)k} - E''$. However, $T_{r,(n+1)k} - E''$ is a failure state and therefore has no paths of length k. Therefore $T_{r,l} - E'$ has no paths of length k and is a failure state.

Thus,

$$\begin{aligned}
|E'| &= \sum_{m=0}^{n-1} (|A_m| + |B_m|) + |A_n^*| \\
&= \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})} + r^{l+1}(r^{-nk-\lfloor (k-1)/2 \rfloor-1} - r^{-l-1}) \\
&= \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor+1}}{1-(r^{-k})} - 1.
\end{aligned}$$

Therefore,

$$\mathrm{CE}_k(T_{r,l}) \le \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor + 1}}{1-(r^{-k})} - 1,$$

which implies

$$\mathrm{CE}_k(T_{r,l}) = \frac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \frac{1-(r^{-k})^{\lfloor l/k \rfloor + 1}}{1-(r^{-k})} - 1.$$

Combining all three of these cases, we see that

$$\mathrm{CE}_k(T_{r,l}) = \begin{cases} 0, & l \le \left\lfloor \frac{1}{2}(k-1) \right\rfloor, \\ \dfrac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \dfrac{1-(r^{-k})^{\lfloor l/k \rfloor}}{1-(r^{-k})}, & nk \le l \le nk + \left\lfloor \frac{1}{2}(k-1) \right\rfloor, \\ \dfrac{r^l}{r^{\lfloor (k-1)/2 \rfloor}} \cdot \dfrac{1-(r^{-k})^{\lfloor l/k \rfloor + 1}}{1-(r^{-k})} - 1, & \text{else,} \end{cases}$$

where n is a positive integer. □

3B2. *General trees.* Although finding a solution for general trees is too difficult, the general principles for perfect r-ary trees will still hold for general trees. Since each edge removal creates a new component, we need to remove edges that create components of diameter less than k which have as large an order as possible. Some bounds could easily be created based on minimum and maximum degree. Other special trees including caterpillar graphs, lobster graphs, and binary trees could be computed using the techniques outlined for the perfect r-ary tree.

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