involve
a journal of mathematics

# On nonabelian representations of twist knots

James C. Dean and Anh T. Tran

msp

2016 vol. 9, no. 5

INVOLVE 9:5 (2016)
dx.doi.org/10.2140/involve.2016.9.831

# On nonabelian representations of twist knots

James C. Dean and Anh T. Tran

(Communicated by Jim Hoste)

We study representations of the knot groups of twist knots into $SL_2(\mathbb{C})$. The set of nonabelian $SL_2(\mathbb{C})$ representations of a twist knot $K$ is described as the zero set in $\mathbb{C}\times\mathbb{C}$ of a polynomial $P_K(x, y) = Q_K(y) + x^2 R_K(y) \in \mathbb{Z}[x, y]$, where $x$ is the trace of a meridian. We prove some properties of $P_K(x, y)$. In particular, we prove that $P_K(2, y) \in \mathbb{Z}[y]$ is irreducible over $\mathbb{Q}$. As a consequence, we obtain an alternative proof of a result of Hoste and Shanahan that the degree of the trace field is precisely two less than the minimal crossing number of a twist knot.

## 1. Introduction

Let $J(k, l)$ be the two-bridge knot/link in Figure 1, where $k, l \neq 0$ denote the numbers of half-twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists. Note that $J(k, l)$ is a knot if and only if $kl$ is even. The knots $J(2, 2n)$, where $n \neq 0$, are known as twist knots. Moreover, $J(2, 2)$ is the trefoil knot and $J(2, -2)$ is the figure-eight knot. For more information about $J(k, l)$, see [Hoste and Shanahan 2004].

We study representations of the knot groups of twist knots into $SL_2(\mathbb{C})$, where $SL_2(\mathbb{C})$ denotes the set of all $2 \times 2$ matrices with determinant 1. From now on we fix a twist knot $J(2, 2n)$. By [Hoste and Shanahan 2001] the knot group of $J(2, 2n)$ has a presentation $\pi_1(J(2, 2n)) = \langle c, d \mid cu = ud\rangle$, where $c, d$ are meridians and $u = (cd^{-1}c^{-1}d)^n$. This presentation is closely related to the standard presentation of the knot group of a two-bridge knot. Note that $J(2, 2n)$ is the twist knot $K_{2n}$ in [Hoste and Shanahan 2001]. In this note we will follow [Tran 2015b, Lemma 1.1] and use a different presentation,

$$\pi_1(J(2, 2n)) = \langle a, b \mid aw = wb\rangle,$$

where $a, b$ are meridians and $w = (ab^{-1})^{-n}a(ab^{-1})^n$. This presentation has been shown to be useful for studying invariants of twist knots; see [Nagasato and Tran 2013; Tran 2013a; 2015a; 2015b].

*MSC2010:* primary 57N10; secondary 57M25.
*Keywords:* Chebychev polynomial, nonabelian representation, parabolic representation, trace field, twist knot.

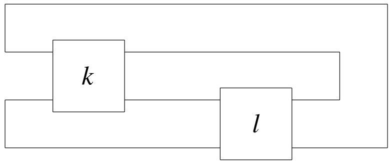


**Figure 1.** The two-bridge knot/link $J(k, l)$.

A representation $\rho : \pi_1(J(2, 2n)) \to \mathrm{SL}_2(\mathbb{C})$ is called nonabelian if the image of $\rho$ is a nonabelian subgroup of $\mathrm{SL}_2(\mathbb{C})$. Suppose $\rho : \pi_1(J(2, 2n)) \to \mathrm{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2-y & s^{-1} \end{bmatrix},$$

where $s \neq 0$ and $y \neq 2$ satisfy a polynomial equation $P_n(s, y) = 0$. The polynomial $P_n$ can be chosen so that $P_n(s, y) = P_n(s^{-1}, y)$, and hence it can be considered as a polynomial in the variables $x := s + s^{-1}$ and $y$. Note that $x = \operatorname{tr} \rho(a) = \operatorname{tr} \rho(b)$ and $y = \operatorname{tr} \rho(ab^{-1})$. An explicit formula for $P_n(x, y)$ will be derived in Section 2 and it is given by

$$P_n(x, y) = 1 - (y + 2 - x^2) S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)),$$

where the $S_k(z)$ are the Chebychev polynomials of the second kind defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = z S_{k-1}(z) - S_{k-2}(z)$ for all integers $k$. Note that $P_n(x, y)$ is different from the Riley polynomial [1984] of the two-bridge knot $J(2, 2n)$; see, e.g., [Nagasato and Tran 2013]. Moreover, $P_n(2, y)$ is also different from the polynomial $\Phi_{-n}(y)$ studied in [Hoste and Shanahan 2001].

In this note we prove the following two properties of $P_n(x, y)$.

**Theorem 1.** *Suppose $x_0^2 \in \mathbb{R}$ such that $4 - 1/|n| < x_0^2 \leq 4$. Then the polynomial $P_n(x_0, y)$ has no real roots $y$ if $n < 0$, and has exactly one real root $y$ if $n > 0$.*

**Theorem 2.** *The polynomial $P_n(2, y) \in \mathbb{Z}[y]$ is irreducible over $\mathbb{Q}$.*

A nonabelian representation $\rho : \pi_1(J(2, 2n)) \to \mathrm{SL}_2(\mathbb{C})$ is called parabolic if the trace of a meridian is equal to 2. The zero set in $\mathbb{C}$ of the polynomial $P_n(2, y)$ describes the set of all parabolic representations of the knot group of $J(2, 2n)$ into $\mathrm{SL}_2(\mathbb{C})$. Theorem 1 is related to the problem of determining the existence of real parabolic representations in the study of the left-orderability of the fundamental groups of cyclic branched covers of two-bridge knots; see [Hu 2015; Tran 2015a].

As in the proof of [Hoste and Shanahan 2001, Theorem 1], Theorem 2 gives an alternative proof of a result of Hoste and Shanahan that the degree of the trace field is precisely two less than the minimal crossing number of a twist knot. Indeed, by definition the trace field of a hyperbolic knot $K$ is the extension field

$\mathbb{Q}(\operatorname{tr}\rho_0(g) : g \in \pi_1(K))$, where $\rho_0 : \pi_1(K) \to \mathrm{SL}_2(\mathbb{C})$ is a discrete faithful representation. The representation $\rho_0$ is a parabolic representation. Since $P_n(2, y)$ is irreducible over $\mathbb{Q}$, the trace field of the twist knot $J(2, 2n)$ is $\mathbb{Q}(y_0)$, where $y_0$ is a certain complex root of $P_n(2, y)$ corresponding to the presentation $\rho_0$. Consequently, the degree of $P_n(2, y)$ gives the degree of the trace field. The conclusion follows, since the minimal crossing number of $J(2, 2n)$ is $2n+1$ if $n > 0$ and is $2-2n$ if $n < 0$.

The rest of this note is devoted to the proofs of Theorems 1 and 2.

## 2. Proofs of Theorems 1 and 2

In this section we first recall some properties of the Chebychev polynomials $S_k(z)$. We then compute the polynomial $P_n(x, y)$. Finally, we prove Theorems 1 and 2.

***Chebychev polynomials.*** Recall that the $S_k(z)$ are the Chebychev polynomials defined by $S_0(z) = 1,\ S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers $k$. Note that $S_k(2) = k + 1$ and $S_k(-2) = (-1)^k(k + 1)$. Moreover, if $z = t + t^{-1}$, where $t \neq \pm 1$, then

$$S_k(z) = \frac{t^{k+1} - t^{-(k+1)}}{t - t^{-1}}.$$

It is easy to see that $S_{-k}(z) = -S_{k-2}(z)$ for all integers $k$.

The following lemma is elementary; see, e.g., [Tran 2013b, Lemma 1.4].

**Lemma 2.1.** *One has*

$$S_k^2(z) - zS_k(z)S_{k-1}(z) + S_{k-1}^2(z) = 1$$

*for all integers $k$.*

**Lemma 2.2.** *For all $k \geq 1$ one has*

$$S_k(z) = \prod_{j=1}^{k}\left(z - 2\cos\frac{j\pi}{k+1}\right),$$

$$S_k(z) - S_{k-1}(z) = \prod_{j=1}^{k}\left(z - 2\cos\frac{(2j-1)\pi}{2k+1}\right).$$

*Proof.* We prove the second formula. The first one can be proved similarly.

Since $S_k(z) - S_{k-1}(z)$ is a polynomial of degree $k$, it suffices to show that its roots are

$$2\cos\frac{(2j-1)\pi}{2k+1},$$

where $1 \leq j \leq k$. Let

$$\theta_j = \frac{(2j-1)\pi}{2k+1}.$$

Then $e^{i(2k+1)\theta_j} = -1$. Hence, if $z = 2\cos\theta_j = e^{i\theta_j} + e^{-i\theta_j}$ then we have

$$S_k(z) = \frac{e^{i(k+1)\theta_j} - e^{-i(k+1)\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = \frac{-e^{-ik\theta_j} + e^{ik\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = S_{k-1}(z).$$

This means that $z = 2\cos\theta_j$ is a root of $S_k(z) - S_{k-1}(z)$. □

**Lemma 2.3.** *Suppose $z \in \mathbb{R}$ such that $-2 \le z \le 2$. Then*

$$|S_{k-1}(z)| \le |k|$$

*for all integers $k$.*

*Proof.* See [Tran 2015a, Lemma 2.6]. □

**Lemma 2.4.** *Suppose $M \in \mathrm{SL}_2(\mathbb{C})$. Then*

$$M^k = S_{k-1}(z)M - S_{k-2}(z)I$$

*for all integers $k$, where $I$ is the $2 \times 2$ identity matrix and $z := \operatorname{tr} M$.*

*Proof.* Since $\det M = 1$, by the Cayley–Hamilton theorem we have $M^2 - zM + I = 0$. This implies that $M^k - zM^{k-1} + M^{k-2} = 0$ for all integers $k$. Then, by induction on $k$ we have $M^k = S_{k-1}(z)M - S_{k-2}(z)I$ for all $k \ge 0$.

For $k < 0$, since $\operatorname{tr} M^{-1} = \operatorname{tr} M = z$ we have

$$\begin{aligned} M^k = (M^{-1})^{-k} &= S_{-k-1}(z)M^{-1} - S_{-k-2}(z)I \\ &= -S_{k-1}(z)(zI - M) + S_k(z)I. \end{aligned}$$

The lemma follows, since $zS_{k-1}(z) - S_k(z) = S_{k-2}(z)$. □

***The polynomial $P_n$.*** Recall that the knot group of $J(2, 2n)$ has the presentation

$$\pi_1(J(2, 2n)) = \langle a, b \mid aw = wb \rangle,$$

where $a, b$ are meridians and $w = (ab^{-1})^{-n} a (ab^{-1})^n$. See [Tran 2015b, Lemma 1.1].

Suppose $\rho : \pi_1(J(2, 2n)) \to \mathrm{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2-y & s^{-1} \end{bmatrix},$$

where $s \ne 0$ and $y \ne 2$ satisfy a polynomial equation $P_n(s, y) = 0$. We now compute the polynomial $P_n$ from the matrix equation $\rho(aw) = \rho(wb)$.

Since

$$\rho(ab^{-1}) = \begin{bmatrix} y-1 & s \\ s^{-1}(y-2) & 1 \end{bmatrix},$$

by Lemma 2.4 we have

$$\begin{aligned}\rho((ab^{-1})^n) &= S_{n-1}(y)\rho(ab^{-1}) - S_{n-2}(y)I \\ &= \begin{bmatrix}(y-1)S_{n-1}(y) - S_{n-2}(y) & sS_{n-1}(y) \\ s^{-1}(y-2)S_{n-1}(y) & S_{n-1}(y) - S_{n-2}(y)\end{bmatrix}.\end{aligned}$$

Hence, by a direct (but lengthy) calculation we have

$$\begin{aligned}\rho(aw) - \rho(wb) &= \rho\big(a(ab^{-1})^{-n}a(ab^{-1})^n\big) - \rho\big((ab^{-1})^{-n}a(ab^{-1})^n b\big) \\ &= \begin{bmatrix}(y-2)P_n(s,y) & sP_n(s,y) \\ -s^{-1}(y-2)P_n(s,y) & 0\end{bmatrix},\end{aligned}$$

where

$$P_n(s,y) = (s^2+s^{-2}+1-y)S_{n-1}^2(y) - (s^2+s^{-2})S_{n-1}(y)S_{n-2}(y) + S_{n-2}^2(y).$$

By Lemma 2.1 we have $S_{n-1}^2(y) - yS_{n-1}(y)S_{n-2}(y) + S_{n-2}^2(y) = 1$. Hence

$$P_n(s,y) = 1 - (y - s^2 - s^{-2})S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)).$$

Since $P_n(s,y) = P_n(s^{-1},y)$, from now on we consider $P_n$ as a polynomial in the variables $x = s + s^{-1}$ and $y$. With these new variables we have

$$P_n(x,y) = 1 - (y + 2 - x^2)S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)).$$

***Proof of Theorem 1.*** We first prove the following lemma.

**Lemma 2.5.** *Suppose $x_0^2 \in \mathbb{R}$ such that $4 - 1/|n| < x_0^2 \le 4$. If $y \in \mathbb{R}$ satisfies $P_n(x_0,y) = 0$, then $y > 2$.*

*Proof.* Since $P_n(x_0,y) = 0$, we have $S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)) = (y+2-x_0^2)^{-1}$. Hence

$$\begin{aligned}\big((y+2-x_0^2)S_{n-1}(y)\big)^{-2} &= (S_{n-1}(y) - S_{n-2}(y))^2 \\ &= 1 + (y-2)S_{n-1}(y)S_{n-2}(y) \\ &= 1 + (y-2)\big(S_{n-1}^2(y) - (y+2-x_0^2)^{-1}\big),\end{aligned}$$

which implies that

$$1 = (y+2-x_0^2)(4-x_0^2)S_{n-1}^2(y) + (y-2)(y+2-x_0^2)^2S_{n-1}^4(y).$$

Assume $y \le 2$. Then it follows from the above equation that

$$1 \le (y+2-x_0^2)(4-x_0^2)S_{n-1}^2(y). \tag{2-1}$$

In particular, $y > x_0^2 - 2 > -2$. Since $-2 < y \le 2$, by Lemma 2.3 we have $S_{n-1}^2(y) \le n^2$. Hence

$$(y+2-x_0^2)(4-x_0^2)S_{n-1}^2(y) \le (4-x_0^2)^2n^2 < 1.$$

This contradicts (2-1). □

We now complete the proof of Theorem 1. Suppose $x_0^2 \in \mathbb{R}$ and $4-1/|n| < x_0^2 \le 4$. By Lemma 2.5, it suffices to consider $P_n(x_0, y)$, where $y$ is a real number greater than 2. The equation $P(x_0, y) = 0$ is equivalent to

$$x_0^2 - 4 = y - 2 - \frac{1}{S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y))}. \tag{2-2}$$

Denote by $f_n(y)$ the right-hand side of (2-2), where $y > 2$. We now use the factorizations of $S_{n-1}(y)$ and $S_{n-1}(y) - S_{n-2}(y)$ in Lemma 2.2.

If $n = -1$ then

$$f_n(y) = y - 2 + \frac{1}{y-1} > 0 \ge x_0^2 - 4.$$

Hence $f_n(y) = x_0^2 - 4$ has no solutions on $(2, \infty)$.

If $n < -1$ then, by letting $m = -n > 1$, we have

$$\begin{aligned} f_n(y) &= y - 2 + \frac{1}{S_{m-1}(y)(S_m(y) - S_{m-1}(y))} \\ &= y - 2 + \frac{1}{\prod_{k=1}^{m-1}\left(y - 2\cos\frac{k\pi}{m}\right)\prod_{l=1}^{m}\left(y - 2\cos\frac{(2l-1)\pi}{2m+1}\right)} > 0 \ge x_0^2 - 4. \end{aligned}$$

Hence $f_n(y) = x_0^2 - 4$ has no solutions on $(2, \infty)$.

If $n = 1$ then $f_n(y) = y - 3$. Since $x_0^2 > 3$, the equation $f_n(y) = x_0^2 - 4$ has a unique solution $y = x_0^2 - 1$ on $(2, \infty)$.

If $n > 1$ then we have

$$f_n(y) = y - 2 - \frac{1}{\prod_{k=1}^{n-1}\left(y - 2\cos\frac{k\pi}{n}\right)\prod_{l=1}^{n-1}\left(y - 2\cos\frac{(2l-1)\pi}{2n-1}\right)}.$$

It is easy to see that $f_n(y)$ is increasing on $(2, \infty)$. Moreover, $\lim_{y\to\infty} f_n(y) = \infty$ and $\lim_{y\to 2} f_n(y) = -1/n < x_0^2 - 4$. Hence $f_n(y) = x_0^2 - 4$ has a unique solution on $(2, \infty)$.

The proof of Theorem 1 is complete.

***Proof of Theorem 2.*** We write $P_n(y)$ for $P_n(2, y)$. Let $y = t^2 + t^{-2}$. Then

$$\begin{aligned} P_n(y) &= (S_{n-1}(y) - S_{n-2}(y))^2 - (y-2)S_{n-1}^2(y) \\ &= \frac{(t^{2n} + t^{2-2n})^2 - t^2(t^{2n} - t^{-2n})^2}{(t^2+1)^2} \\ &= \frac{(t^{2n} + t^{2-2n} + t^{2n+1} - t^{1-2n})(t^{2n} + t^{2-2n} - t^{2n+1} + t^{1-2n})}{(t^2+1)^2}. \end{aligned}$$

Up to a factor $t^k$, each of $t^{2n} + t^{2-2n} + t^{2n+1} - t^{1-2n}$ and $t^{2n} + t^{2-2n} - t^{2n+1} + t^{1-2n}$ is obtained from the other by replacing $t$ by $t^{-1}$. To show that $P_n(y)$ is irreducible over $\mathbb{Q}$, it suffices to show that

$$t^{4n}+t^{4n-1}+t-1=(t^2+1)Q_n(t), \tag{2-3}$$

where $Q_n(t) \in \mathbb{Z}[t]$ is irreducible over $\mathbb{Q}$.

As in the proof of [Baker and Petersen 2013, Lemma 6.8], we will use the following theorem of Ljunggren [1960]. Consider a polynomial of the form $R(t)=t^{k_1}+\varepsilon_1 t^{k_2}+\varepsilon_2 t^{k_3}+\varepsilon_3$, where $\varepsilon_j=\pm 1$ for $j=1,2,3$. Then, if $R$ has $r>0$ roots of unity as roots then $R$ can be decomposed into two factors, one of degree $r$ which has these roots of unity as zeros and the other which is irreducible over $\mathbb{Q}$. Hence, to prove (2-3) it suffices to show that $\pm i$ are the only roots of unity which are roots of $t^{4n}+t^{4n-1}+t-1$ and these occur with multiplicity 1.

Let $t$ be a root of unity such that $t^{4n}+t^{4n-1}+t-1=0$. Write $t=e^{i\theta}$, where $\theta \in \mathbb{R}$. Since $t^{2n-1}+t^{1-2n}+t^{2n}-t^{-2n}=0$, we have

$$2\cos(2n-1)\theta+2i\sin 2n\theta=0,$$

which implies that both $\cos(2n-1)\theta$ and $\sin 2n\theta$ are equal to zero. There exist integers $k, l$ such that $(2n-1)\theta=\left(k+\frac{1}{2}\right)\pi$ and $2n\theta=l\pi$. This implies that $(2k+1)/l=(2n-1)/n$. Since $(2n-1)/n$ is a reduced fraction, there exists an *odd* integer $m$ such that $2k+1=m(2n-1)$ and $l=mn$. Hence $\theta=\frac{1}{2}m\pi$, which implies that $t=e^{i\theta}=\pm i$. It is easy to verify that $\pm i$ are roots of $t^{4n}+t^{4n-1}+t-1=0$ with multiplicity 1.

Ljunggren's theorem then completes the proof of Theorem 2.

## Acknowledgements

This research was supported by the Summer 2015 Pioneer REU in the Department of Mathematical Sciences at the University of Texas at Dallas. We would like to thank the referee for carefully reading our paper and for giving helpful comments and suggestions.

## References

[Baker and Petersen 2013] K. L. Baker and K. L. Petersen, "Character varieties of once-punctured torus bundles with tunnel number one", *Internat. J. Math.* **24**:6 (2013), Article ID #1350048. MR 3078072 Zbl 1275.57016

[Hoste and Shanahan 2001] J. Hoste and P. D. Shanahan, "Trace fields of twist knots", *J. Knot Theory Ramifications* **10**:4 (2001), 625–639. MR 1831680 Zbl 1003.57014

[Hoste and Shanahan 2004] J. Hoste and P. D. Shanahan, "A formula for the A-polynomial of twist knots", *J. Knot Theory Ramifications* **13**:2 (2004), 193–209. MR 2047468 Zbl 1057.57010

[Hu 2015] Y. Hu, "Left-orderability and cyclic branched coverings", *Algebr. Geom. Topol.* **15**:1 (2015), 399–413. MR 3325741 Zbl 06425407

[Ljunggren 1960] W. Ljunggren, "On the irreducibility of certain trinomials and quadrinomials", *Math. Scand.* **8** (1960), 65–70. MR 0124313 Zbl 0095.01305

[Nagasato and Tran 2013] F. Nagasato and A. T. Tran, "Some families of minimal elements for a partial ordering on prime knots", preprint, 2013. To appear in *Osaka J. Math.* arXiv 1301.0138

[Riley 1984] R. Riley, "Nonabelian representations of 2-bridge knot groups", *Quart. J. Math. Oxford Ser.* (2) **35**:138 (1984), 191–208. MR 745421 Zbl 0549.57005

[Tran 2013a] A. T. Tran, "On the twisted Alexander polynomial for representations into $SL_2(\mathbb{C})$", *J. Knot Theory Ramifications* **22**:10 (2013), Article ID #1350059. MR 3125898 Zbl 1278.57023

[Tran 2013b] A. T. Tran, "The universal character ring of some families of one-relator groups", *Algebr. Geom. Topol.* **13**:4 (2013), 2317–2333. MR 3073918 Zbl 1280.57014

[Tran 2015a] A. T. Tran, "On left-orderability and cyclic branched coverings", *J. Math. Soc. Japan* **67**:3 (2015), 1169–1178. MR 3376583 Zbl 1328.57017

[Tran 2015b] A. T. Tran, "On left-orderable fundamental groups and Dehn surgeries on knots", *J. Math. Soc. Japan* **67**:1 (2015), 319–338. MR 3304024 Zbl 06422613

Received: 2015-07-06 Revised: 2015-10-30 Accepted: 2015-11-03

jdean@udallas.edu *University of Dallas, Irving, TX 75062, United States*

att140830@utdallas.edu *Department of Mathematical Sciences, The University of Texas at Dallas, Richardson, TX 75080, United States*

# involve

msp.org/involve

## INVOLVE YOUR STUDENTS IN RESEARCH

*Involve* showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.



See inside back cover or msp.org/involve for submission instructions. The subscription price for 2016 is US $160/year for the electronic version, and $215/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

# involve

2016 vol. 9 no. 5