Polygonal bicycle paths
and the Darboux transformation

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(Communicated by Kenneth S. Berenhaut)

A bicycle (n, k)-gon is an equilateral n-gon whose k diagonals are of equal length. In this paper we introduce periodic bicycle (n, k)-paths, which are a natural variation in which the polygon is replaced with a periodic polygonal path, and study their rigidity and integrals of motion.

1. Background

Our motivation comes from three seemingly unrelated problems. The first is the problem of *floating bodies of equilibrium in two dimensions*. From 1935 to 1941, mathematicians at the University of Lviv, among them Stefan Banach and Mark Kac, collected mathematical problems in a book, which became known as "the Scottish book", since they often met in the Scottish Coffee House. Stanislaw Ulam posed problem 19 of this book: "Is a sphere the only solid of uniform density which will float in water in any position?" The answer in the two-dimensional case, as it turns out, depends on the density of the solid.

The second problem, known as the *tire track problem*, originated in the story, "The adventure of the priory school" by Arthur Conan Doyle, where Sherlock Holmes and Dr. Watson discuss in view of the two tire tracks of a bicycle which way the bicycle went. The problem is: "Is it possible that tire tracks other than circles or straight lines are created by bicyclists going in both directions?" As shown in Figure 1, the answer to this subtle question is affirmative.

The third problem is that of describing the trajectories of *electrons in a parabolic magnetic field*. All three problems turn out to be equivalent [Wegner 2007].

Often in mathematics it is fruitful to discretize a problem. As such, S. Tabachnikov [2006] proposed a "discrete bicycle curve" (also known as a "bicycle polygon"), which is a polygon satisfying discrete analogs of the properties of a bicycle track. The main requirement turns out to be that, in the language of discrete differential geometry, the polygon is "self-Darboux". That is, the discrete differential
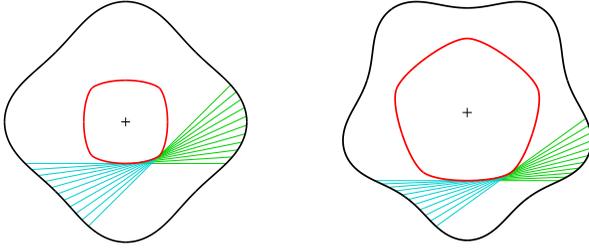

Figure 1. Ambiguous bicycle tracks. The rear-wheel track is the inner curve and the front-wheel track is the outer curve. One cannot tell which way the bicycle went because a bicycle could have followed either one of two trajectories [Wegner 2007].

geometric notion of a discrete Darboux transformation [Bobenko and Suris 2008; Tsuruga 2010], which relates one polygon to another, relates a discrete bicycle curve to itself.

The topic of bicycle curves and polygons belongs to a number of active areas of research. On the one hand, it is part of rigidity theory. As an illustration, R. Connelly and B. Csikós [2009] consider the problem of classifying first-order flexible regular bicycle polygons. Other work on the rigidity theory of bicycle curves and polygons can be found in [Csikós 2007; Cyr 2012; Tabachnikov 2006].

The topic is also part of the subject of discrete integrable systems. This point of view is taken in [Tabachnikov and Tsukerman 2013], where the authors find integrals of motion (i.e., quantities which are conserved) of bicycle curves and polygons under the Darboux transformation and recutting of polygons [Adler 1993; 1995].

In this paper, in analogy with bicycle polygons, we introduce a new concept called a periodic discrete bicycle path and study both its rigidity and integrals.

2. Bicycle (n, k)-paths

A bicycle (n, k)-gon is an equilateral n-gon whose k diagonals are of equal length [Tabachnikov 2006]. We consider the following analog.

Definition 1. Define $P = \{V_i \in \mathbb{R}^2 : i \in \mathbb{Z}\}$ (for brevity, $V_0 V_1 \cdots V_{n-1}$) to be a *discrete periodic bicycle* (n, k)-*path* (or discrete (n, k)-path) if the following conditions hold:

(i) $V_{n+i} = V_i + e_1$ for all i, where $e_1 = (1, 0)$ and $V_0 = (0, 0)$ (periodicity condition).

(ii) $|V_i V_{i+1}| = |V_j V_{j+1}|$ for all i, j (equilateralness).

(iii) $|V_i V_{i+k}| = |V_j V_{j+k}|$ for all i, j (equality of k-diagonals).

Definition 1 is meant to model the motion of a bicycle whose trajectory is spatially periodic. The condition that $|V_j V_{j+1}|$ is independent of j prescribes a constant speed for the motion of the bike. The condition that $|V_j V_{j+k}|$ is independent of j

represents the ambiguity of the direction in which the bicycle went (see [Tabachnikov 2006] for details).

Some natural questions regarding periodic (n, k)-paths are for which pairs (n, k) they exist, how many there are and whether they are rigid or flexible. We consider these questions in Section 3. A simple example of a bicycle (n, k)-path, analogous to the regular (n, k)-polygon, is $V_i = (i/n, 0)$, i.e., when all vertices lie at equal intervals on the line. We call this the *regular path*. Since bicycle (n, k)-paths are discretized bicycle paths, it is also interesting to see if there are any integrals of motion. We show that this is indeed the case in Section 4, by showing that area is an integral of motion.

3. Rigidity

The following two lemmas will be helpful when analyzing the rigidity of discrete bicycle paths.

Lemma 2. *Let $n \in \mathbb{N}$, $\chi_i \in \{-1, 1\}$ for every $i \in \mathbb{Z}/n\mathbb{Z}$ and let*

$$S = \{(x_0, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n : (x_{i+1} - x_i)^2 = (x_{j+1} - x_j)^2 \text{ for all } i, j \in \mathbb{Z}/n\mathbb{Z}\}.$$

Then

$$S = \left\{(x_0, x_1, \ldots, x_{n-1}) : x_{i+1} = x_i + \chi_i r \text{ for } i \in \mathbb{Z}/n\mathbb{Z}, \sum_{i=0}^{n-1} r \chi_i = 0 \text{ and } r \geq 0\right\}.$$

In particular, if n is odd, then $S = \{(x_0, x_1, \ldots, x_{n-1}) : x_i = x_j \text{ for all } i, j \in \mathbb{Z}/n\mathbb{Z}\}$.

Proof. First note that the candidate set is well-defined since

$$x_{j+n} = x_j + \sum_{i=j}^{j+n-1} r \chi_i = x_j + \sum_{i=0}^{n-1} r \chi_i = x_j.$$

Let $(x_0, x_1, \ldots, x_{n-1}) \in S$. Recall that

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

and that $\text{sgn}(x)|x| = x$. Set $r := |x_{i+1} - x_i|$ and $\chi_i = \text{sgn}(x_{i+1} - x_i) + (1 - \text{sgn}(r))$. Then

$$x_{i+1} = x_i + \chi_i r,$$

and

$$\sum_{i=0}^{n-1} r \, \text{sgn}(x_{i+1} - x_i) = 0.$$

It follows that any n-tuple in S satisfies the conditions $x_{i+1} = x_i + \chi_i r$, $\sum_{i=0}^{n-1} r \chi_i = 0$ and $r \geq 0$. The opposite inclusion is clear. $\qquad \square$

Lemma 3. *Let $x_i \in \mathbb{R}$ for every $i \in \mathbb{Z}$ with $x_0 = 0$ and let k and n be coprime integers. Assume that $x_{i+k} - x_i = x_i - x_{i-k}$ for each i and that $x_{i+n} = 1 + x_i$. Then $x_i = i/n$ for each i.*

Proof. Define z_i via $x_i = z_i + i/n$. The hypothesis $x_{i+n} = 1 + x_i$ implies that

$$z_{i+n} = z_i.$$

The difference

$$\Delta z := z_{i+k} - z_i$$

is independent of i due to the assumption that $x_{i+k} - x_i = x_i - x_{i-k}$ and because k and n are coprime. This implies that

$$0 = z_{i+n} - z_i = z_{i+nk} - z_i = n\Delta z.$$

It follows that $z_i = 0$ for every i. $\square$

The following theorem gives a classification of a family of periodic (n, k)-paths.

Theorem 4. *The discrete $(n, dn - 1)$-paths $V_i = (x_i, y_i)$, $i \in \mathbb{N}$ with $d \neq 0$ are exactly those paths which satisfy*

$$x_j = \frac{j}{n}$$

and

$$y_{j+1} = y_j + \chi_j r \quad \text{for } j \in \mathbb{Z}/n\mathbb{Z} \text{ with } \sum_{i=0}^{n-1} r\chi_i = 0 \text{ and } r \geq 0$$

for each j. In particular, if n is odd then a discrete $(n, dn - 1)$-path must be regular.

Proof. For every i,

$$|V_i V_{i+1}| = |V_{i+dn-1} V_{i+dn}|,$$
$$|V_i V_{i+dn-1}| = |V_{i+1} V_{i+dn}|.$$

Therefore

$$(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2,$$
$$(d + x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2 = (d + x_i - x_{i+1})^2 + (y_i - y_{i+1})^2.$$

It follows that

$$d(x_{i+1} - x_i) = d(x_i - x_{i-1}).$$

Since $d \neq 0$,

$$x_{i+1} - x_i = x_i - x_{i-1}.$$

By Lemma 3, $x_j = j/n$ for each j. Now equation $|V_i V_{i+1}| = |V_j V_{j+1}|$ for all i, j implies that

$$(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = (x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2 \quad \text{for all } i, j,$$
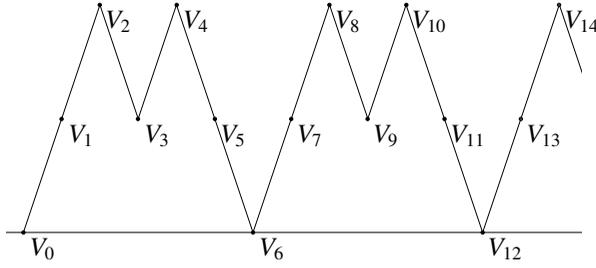
Figure 2. An example of a discrete $(6, 5)$-path.

so that

$$(y_{i+1} - y_i)^2 = (y_{j+1} - y_j)^2.$$

By Lemma 2, we are done.   □

Theorem 5. *The discrete $(n, dn + 1)$-paths $V_i = (x_i, y_i)$, $i \in \mathbb{N}$ with $d \neq 0$ are exactly those paths which satisfy*

$$x_j = \frac{j}{n}$$

and

$$y_{j+1} = y_j + \chi_j r \quad \text{for } j \in \mathbb{Z}/n\mathbb{Z} \text{ with } \sum_{i=0}^{n-1} r\chi_i = 0 \text{ and } r \geq 0$$

for each j. In particular, if n is odd then a discrete $(n, dn+1)$-path must be regular.

Proof. Set $C_1 = |V_i V_{i+dn+1}|^2$ and $C_2 = |V_i V_{i+1}|^2$. Then

$$(d + x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = C_1,$$
$$(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = C_2.$$

Substituting, we get

$$d^2 + 2d(x_{i+1} - x_i) + C_2 = C_1,$$

so that $x_{i+1} - x_i$ is constant. By Lemma 3, $x_i = i/n$. It follows that $(y_{i+1} - y_i)^2$ is constant, so by Lemma 3 we are done.   □

Corollary 6. *Any $(n, dn + 1)$-path is an $(n, dn - 1)$-path and vice versa.*

For an example, see Figure 2.

4. Darboux transformation and integrals

It is important to make a distinction between infinitesimal "trapezoidal" movement and infinitesimal "parallelogram" movement of the bicycle. Consider a pair of conjoined bikes, sharing a back wheel and facing in opposite directions. Since

this bicycle moves in such a way that the distance between the turnable wheels is constant, at each moment of time the turnable wheels must enclose equal angles with the line of the frame. When the two turnable wheels are parallel, the trike is gliding, but then the common back wheel of the bikes is slipping, which is not allowed. That is why we exclude parallelogram movements from consideration for the remainder of this paper.

Definition 7 (trapezoidal condition). We will say that a discrete (n, k)-path satisfies the trapezoidal condition if $V_i V_{i+k+1}$ and $V_{i+1} V_{i+k}$ are parallel for each $i \in \mathbb{Z}$.

As an illustration of these concepts, consider Figure 2: $V_0 V_1 V_5 V_6$ is a trapezoidal motion, while $V_1 V_2 V_6 V_7$ is a parallelogram motion. Consequently, the bicycle path in the figure does not satisfy the trapezoidal condition.

Assuming the trapezoidal condition, we may view bicycle paths in terms of an important construction in discrete differential geometry called the Darboux transformation [Bobenko and Suris 2008; Tsuruga 2010].

Definition 8 (Darboux transform). We say that two polygons $P = P_1 P_2 \cdots$ and $Q = Q_1 Q_2 \cdots$ are in Darboux correspondence if for each $i = 1, 2, \ldots$, we have that Q_{i+1} is the reflection of P_i in the perpendicular bisector of the segment $P_{i+1} Q_i$.

If segment $P_1 Q_1$ is of length ℓ then for each i, $P_i Q_i$ is of length ℓ. We then say that P and Q are in Darboux correspondence with parameter ℓ. We also note that each quadrilateral $P_i Q_i P_{i+1} Q_{i+1}$ is an isosceles trapezoid.

We denote the map taking vertex P_i to Q_i by $\mathfrak{D}$. We will also refer to the map of polygons $\mathfrak{D}(P) = Q$ by the same letter, since no confusion ought to occur.

Consider a polygonal line P with vertices $V_0, V_1, \ldots, V_{n-1}$. Let v_0 be a vector with its origin at V_0. Having a vector v_i at vertex V_i, we obtain a vertex v_{i+1} of the same length at V_{i+1} via the trapezoidal condition. For example, in Figure 3, $v_1 = P_1 Q_1$ and $v_2 = P_2 Q_2$. For a fixed length of v_0, we may view the map taking v_0 to v_j as a self-map of the circle of radius $|v_0| = |v_j|$ by identifying the circle at V_0 with circle at V_j via parallel translation.

Definition 9 (monodromy map of the Darboux transformation). The monodromy map is the map acting on the identified circles at V_0 and V_n which takes v_0 to v_n.

It is known that the monodromy map is a cross-ratio preserving transformation (in terms of affine coordinates, a fractional linear transformation) on a circle of fixed radius after we identify the circle with the real projective line via stereographic projection [Tabachnikov and Tsukerman 2013]. We will assume throughout, unless otherwise stated, that the monodromy map is acting on a fixed point; in other words, we will assume that the Darboux transform has been chosen so that the initial vector v_0 is equal to the vector v_n, where n is the period. This is analogous to applying the Darboux transform to a closed polygon and requiring that its image is closed also.
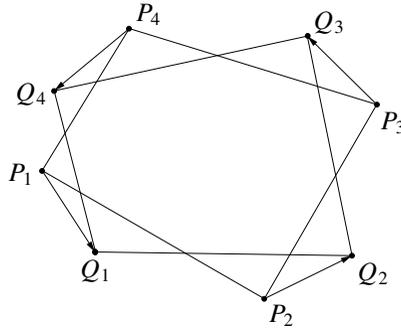
Figure 3. Two polygons in Darboux correspondence.

We mention in passing that in the case of closed polygons, Darboux correspondence implies that the monodromies of the two polygons are conjugated to each other. The invariants of the conjugacy class of the monodromy, viewed as functions of the length parameter, are consequently integrals of the Darboux correspondence [Tabachnikov and Tsukerman 2013].

Connection between Darboux transformation and discrete (n, k)-paths. A discrete (n, k)-path satisfying the trapezoidal condition may be interpreted in terms of the Darboux transform. Indeed, given such a path, we consider the periodic equilateral linkages $L_i = \cdots V_{0+i} V_{k+i} V_{2k+i} \cdots$ for $i = 0, 1, \ldots, k - 1$. The trapezoidal condition implies that there is a Darboux correspondence $\mathfrak{D}(L_i) = L_{i+1}$ of the same parameter (since the (n, k)-path is equilateral) for consecutive linkages (see Figure 4).

The Darboux transformation also preserves the area of periodic paths. More precisely, let $y = -c$ for $c > 0$ sufficiently large so that the periodic path P and its Darboux transformation P' lie completely above $y = -c$. We define an area function as follows. Let $\check{V}_i = (x(V_i), -c)$. We define the area of P to be the signed area of the polygon $\check{V}_0 V_0 V_1 \cdots V_n \check{V}_n$ and denote it by $|P|$. We show that this area is preserved under Darboux transformation (see Figure 5). In particular, it will follow that the area of $V_0 V_k \cdots V_{nk}$ is equal to the area of $V_m V_{k+m} \cdots V_{nk+m}$ for every $m \in \mathbb{Z}$.

Theorem 10. *The Darboux transformation is area-preserving on periodic polygonal paths.*

Proof. Let P and P' be two periodic polygonal paths in Darboux correspondence. We show that the difference of the areas of P and P' is zero. We denote the vertex of P' which corresponds via the Darboux transformation to the vertex V_i in P by V_i' for each i. We have

$$|P| = \sum_{i=0}^{n-1} |\check{V}_i V_i V_{i+1} \check{V}_{i+1}|,$$
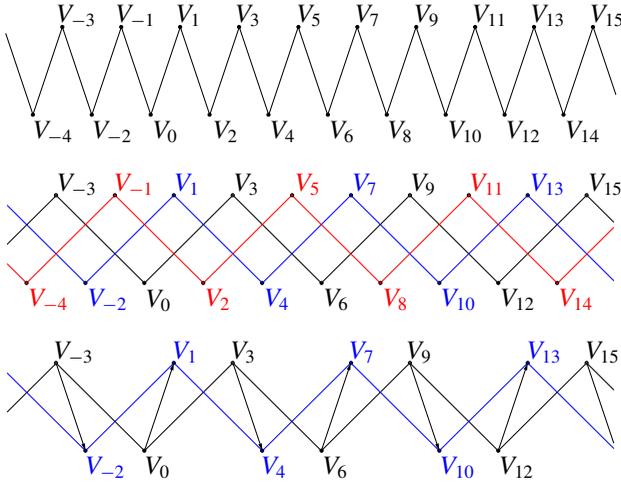
Figure 4. Viewing a discrete (n, k)-path satisfying the trapezoidal condition (top) in terms of the Darboux transformation. The path is decomposed into equilateral linkages (middle). Any two consecutive linkages are in Darboux correspondence (bottom).

and similarly for P'. Therefore

$$|P| - |P'| = \sum_{i=0}^{n-1} |\check{V}_i V_i V_{i+1} \check{V}_{i+1}| - |\check{V}'_i V'_i V'_{i+1} \check{V}'_{i+1}|.$$

From the isosceles trapezoids,

$$|V_i V_{i+1} V'_{i+1}| = |V'_i V'_{i+1} V_i|. \tag{4-1}$$

Also,

$$|\check{V}_i V_i V_{i+1} \check{V}_{i+1}| = |\check{V}_i V_i V'_{i+1} \check{V}'_{i+1}| + |\check{V}'_{i+1} V'_{i+1} V_{i+1} \check{V}_{i+1}| + |V_i V_{i+1} V'_{i+1}|.$$

Similarly,

$$|\check{V}'_i V'_i V'_{i+1} \check{V}'_{i+1}| = |\check{V}'_i V'_i V_i \check{V}_i| + |\check{V}_i V_i V'_{i+1} \check{V}'_{i+1}| + |V'_i V'_{i+1} V_i|.$$

Using (4-1),

$$|\check{V}_i V_i V_{i+1} \check{V}_{i+1}| - |\check{V}'_i V'_i V'_{i+1} \check{V}'_{i+1}| = |\check{V}'_{i+1} V'_{i+1} V_{i+1} \check{V}_{i+1}| - |\check{V}'_i V'_i V_i \check{V}_i|.$$

It follows that

$$|P| - |P'| = \sum_{i=0}^{n-1} |\check{V}'_{i+1} V'_{i+1} V_{i+1} \check{V}_{i+1}| - |\check{V}'_i V'_i V_i \check{V}_i|,$$
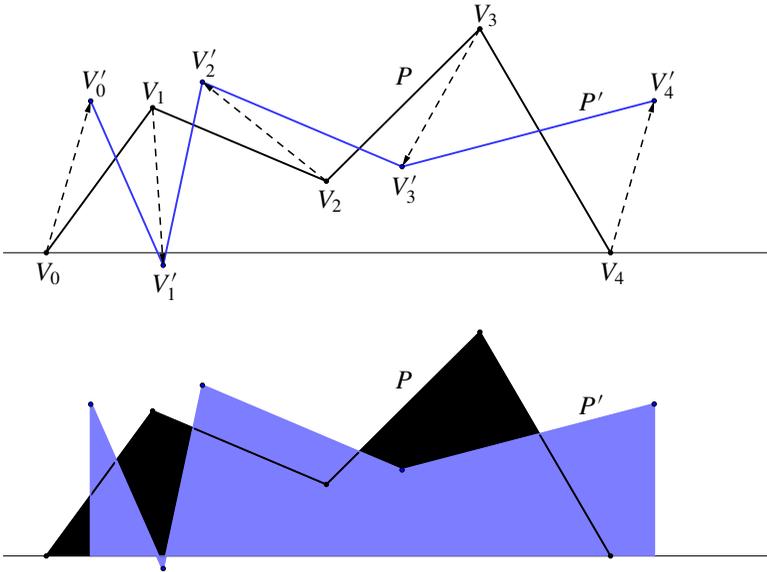
Figure 5. Two periodic paths P and P' in Darboux correspondence. By Theorem 10, the two paths have equal areas under the curve.

which telescopes to

$$|P| - |P'| = |\check{V}_n' V_n' V_n \check{V}_n| - |\check{V}_0' V_0' V_0 \check{V}_0|.$$

Since $V_n' = V_0' + e_1$ and $V_n = V_0 + e_1$, it follows that $\overrightarrow{V_n V_n'} = \overrightarrow{V_0 V_0'}$ and $|\check{V}_n' V_n' V_n \check{V}_n| = |\check{V}_0' V_0' V_0 \check{V}_0|$, so that $|P| = |P'|$. $\qquad \square$

5. Questions

We end our discussion with some research topics and questions of interest concerning bicycle (n, k)-paths.

(1) Construct interesting families of bicycle (n, k)-paths. For example, ones for which the condition $x_j = j/n$ does not hold.

(2) What is the m-th order ($m \in \mathbb{N}$) infinitesimal rigidity theory of bicycle (n, k)-paths like?

(3) For closed bicycle polygons, there are many integrals of motion [Tabachnikov and Tsukerman 2013]. For example, a geometric center called the circumcenter of mass [Tabachnikov and Tsukerman 2014] is invariant under Darboux transformation for closed polygons. Are there other integrals of motion for bicycle (n, k)-paths?

Acknowledgments

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