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Fibonacci Nim and a full characterization of winning moves

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In this paper we will fully characterize all types of winning moves in the "take-away" game of Fibonacci Nim. We prove the known winning algorithm as a corollary of the general winning algorithm and then show that no other winning algorithms exist. As a by-product of our investigation of the game, we will develop useful properties of Fibonacci numbers. We conclude with an exploration of the probability that unskilled player may beat a skilled player and show that as the number of tokens increase, this probability goes to zero exponentially.

1. Introduction

We begin with a brief introduction to the idea of "take-away" games. Schwenk [1970] defined *take-away* games to be a two-person game in which the players alternately diminish an original stock of tokens subject to various restrictions, with the player who removes the last token being the winner.

In the generalized *take-away* game, $\tau(k) = \eta(k-1) - \eta(k)$ where $\eta(k)$ is the number of tokens remaining after the k-th turn so that $\tau(k)$ is the number of tokens removed on the k-th turn. Additionally, for all $k \in \mathbb{N}$, $k \neq 1$, we have $\tau(k) \leq m_k$, where m_k is some function of $\tau(k-1)$. Specifically in Fibonacci Nim, we have $m_k = 2\tau(k)$ for $k > 1$. We will immediately move away from this notation and develop additional notation as it is required. We provide a simple example to familiarize the reader with the game.

Example 1. Let $n = 10$. Player one may remove 1 to 9 tokens. Suppose player one removes 3 tokens. Then, player two may now remove 1 through $2(3) = 6$ tokens. Play continues until one of the players removes the last token.

We will rely heavily on results from [Lekkerkerker 1952], specifically the *Zeckendorff representation* of natural numbers as a sum of Fibonacci numbers.

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The Fibonacci numbers are the positive integers generated by the recursion $F_k = F_{k-1} + F_{k-2}$, where $F_1 = 1 = F_2$ and $k \in \mathbb{N}$. Let $F = \{F_2, F_3, \dots, F_k, \dots\} = \{1, 2, 3, 5, \dots\}$. This is the subset of Fibonacci numbers we will reference throughout this paper. We now present the Zeckendorf representation theorem (ZRT) without proof. A proof of this theorem may be found in [Hoggatt et al. 1973].

Theorem 2 (Zeckendorff representation theorem). *Let $n \in \mathbb{N}$. For $i, r \in \mathbb{N}$ we have $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$, where $i_r - (r-1) > i_{r-1} - (r-2) > \cdots > i_2 - 1 > i_1 \geq 2$. Further, this representation is unique.*

In other words, every positive integer can be written uniquely as a sum of nonconsecutive Fibonacci numbers. Clearly, in the notation of the theorem, $F_{i_r} > F_{i_{r-1}} > \cdots > F_{i_1}$. We will refer to the Zeckendorff Representation theorem frequently, so we abbreviate it by ZRT.

Example 3. $12 = (1)F_6 + (0)F_5 + (1)F_4 + (0)F_3 + (1)F_2 = 8 + 3 + 1.$

Corollary 4. *If $F_{k+1} > n \geq F_k$, then F_k is the largest number in the Zeckendorff representation of n.*

Proof. If $F_{k+1} > n \geq F_k$ then by Zeckendorff's theorem we can write $(n - F_k) = F_d + \cdots + F_{i_1}$. We claim $k - 1 > d$. Suppose not; then $d \geq (k-1)$, thus

$$n = F_k + F_d + \cdots + F_{i_1} \geq F_k + F_d \geq F_k + F_{k-1} = F_{k+1}.$$

However, $F_{k+1} > n \geq F_{k+1}$ is a contradiction. Thus, $k - 1 > d$ so that

$$n = F_k + F_d + \cdots + F_{i_1}$$

is a valid, and thus the only, representation of n by the ZRT. □

The corollary above shows that for any $n \in \mathbb{N}$ where $F_{k+1} > n \geq F_k$, the Zeckendorff representation of n must contain F_k. Therefore, we iteratively may take the maximal Fibonacci number less than n, say F_k, subtract it from n which yields $n - F_k = n' = F_{i_{r'}} + F_{i_{r-1'}} + \cdots + F_{i_{1'}}$ and repeat this process to find each Fibonacci number in the representation of the original number, n.

Definition 5. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ where $r, i, n \in \mathbb{N}$. We define $T(n) = F_{i_1}$. That is, $T(n)$ is the smallest number in the Zeckendorff representation.

Definition 6. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ where $r, i, n, j \in \mathbb{N}$. We now define the *length j tail* to be the specific sum of j consecutive* Fibonacci numbers in the Zeckendorff representation of n beginning with the smallest number, F_{i_1}. We set $T_1(n) = T(n)$ for consistency. Then, $T_j(n) = T(n) + T_{i-1}(n - T(n))$.

The "consecutive*" in Definition 6 refers to the subscripts i_j, i_{j+1} for some $j \in \mathbb{N}$. By the above definitions, we see that the *length j tail* of n is $T_j(n) = F_{i_j} + F_{i_{j-1}} + \cdots + F_{i_1}$ where $r \geq j \geq 1$.

Example 7. Consider

$$\begin{aligned} 33 &= F_8 + F_6 + F_4 + F_2 = 21 + 8 + 3 + 1, \\ 12 &= F_6 + F_4 + F_2 = 8 + 3 + 1. \end{aligned}$$

Then, the *length 3 tails* are

$$\begin{aligned} T_3(33) &= F_6 + F_4 + F_2 = 8 + 3 + 1, \\ T_3(12) &= F_6 + F_4 + F_2 = 8 + 3 + 1. \end{aligned}$$

Hence, 33 and 12 have the same length 3 tail.

Remark 8. By the definition of a length j tail, if $T_j(n) = T_j(m)$, then for any $j \geq s \geq 1$, we have $T_s(n) = T_s(m)$.

Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ be the Zeckendorff representation where

$$F_{i_r} > F_{i_{r-1}} > \cdots > F_{i_1}.$$

Suppose there are n tokens in the pile during the current turn. The known winning algorithm for Fibonacci Nim has the current player take the length 1 tail of n. That is, the player removes $T(n) = F_{i_1}$ tokens. We will prove that this is a winning algorithm in the next section.

In what follows, we will extend the known winning algorithm to include tails that satisfy certain criteria for some given n. We then will prove that this is a complete collection of winning moves and that no others exist. We end this paper by introducing a *losing position strategy* and then derive an upperbound on the probability that an unskilled player may beat a skilled player.

2. Fibonacci Nim strategy

We begin this section by discussing how to win Fibonacci Nim. In the remainder of this paper, we use $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ with $n, r, i \in \mathbb{N}$ as the Zeckendorff representation for some n.

Assume there are n tokens in a given turn which the player whose turn it is may remove from. Let $2p$ denote the maximum number of tokens this player may remove from the n tokens. We can denote this position by $(n, 2p)$. Note, this implies that the previous player removed precisely p tokens.

Definition 9. A *losing position* is such that given the position $(n, 2p)$, $T(n) > 2p$. A *winning position* is any nonlosing position. A *winning move* is such that it results in the next position being a losing position. A *losing move* is any nonwinning move.

We see by Definition 9 that we always want to leave our opponent in a losing position where $T(n) > 2p$. That is, a position where our opponent cannot remove any length j tail, $T_j(n)$. As an immediate consequence, if our opponent cannot remove a tail of n, certainly he cannot remove all of n to win since $n \geq T_j(n) \geq T(n) > 2p$. Therefore, if we can successively give our opponent a losing position, we can ensure we win.

Lemma 10. *For every $i \in \mathbb{N}$ where $i \geq 3$, $2F_{i-1} \geq F_i$ and $F_{i+1} > 2F_{i-1}$.*

Proof. We have $2(F_2) = 2(1) = 2 = F_3$ and $F_4 = 3 > 2 = 2(1) = 2(F_2)$. Assume that $2F_{i-1} \geq F_i$ and $F_{i+1} > 2F_{i-1}$. We have $2(F_i) = 2(F_{i-1} + F_{i-2}) \geq 2F_{i-1} + F_{i-2} = F_i + F_{i-1} = F_{i+1}$ since for each for $j \in \mathbb{N}$, $F_j \geq 1$. Similarly,

$$\begin{aligned} F_{i+2} = F_{i+1} + F_i &= (F_i + F_{i-1}) + (F_{i-1} + F_{i-2}) \\ &> F_i + (F_{i-1} + F_{i-2}) = 2F_i. \end{aligned}$$ □

Lemma 11 below implies that if on a given turn our opponent has a losing position to play from, regardless of how he plays, our next play will be from a winning position.

Lemma 11. *Let $n \in \mathbb{N}$. For any p with $T(n) > p$, $(n - p, 2p)$ is a winning position.*

Proof. Let $n \in \mathbb{N}$. Assume $T(n) > p$. We have, $n - p = F_{i_r} + \cdots + F_{i_1} - p$. Define $m = T(n) - p = F_{i'_r} + \cdots + F_{i'_1}$. Suppose $(n - p, 2p)$ is a losing position. Then, $T(n - p) > 2p$ and by Lemma 10, $2F_{i'_1-1} \geq F_{i'_1} > 2p$. Hence, the Zeckendorff representation of p does not include $F_{i'_1-1}$, thus $p = F_{i''_r} + \cdots + F_{i''_1}$ where $F_{i'_1-1} > F_{i''_r}$. But then, $n = F_{i_r} + \cdots + F_{i_2} + F_{i'_r} + \cdots + F_{i'_1} + F_{i''_r} + \cdots + F_{i''_1}$ is a valid Zeckendorff representation of n. This is a contradiction since Zeckendorff representations are unique. Hence, we must have $2p \geq T(n - p)$. Since $F_{i_2} > T(n) > T(n) - p$, then, $n - p = F_{i_r} + \cdots + F_{i_2} + m$ is a valid Zeckendorff representation of $n - p$ and hence the only representation. Thus, the next position, $(n - p, 2p)$ has $2p \geq T(n - p)$ so that $(n - p, 2p)$ is a winning position. □

Lemma 12 below paired with Lemma 11 proves the known winning strategy. That is, if we take the length 1 tail of n, $T(n)$, the next position is a losing position. Successively implementing this lemma results in winning the game in a finite number of moves.

Lemma 12. *Let $n \in \mathbb{N}$. Set $p = T(n)$. Then $(n - p, 2p)$ is a losing position.*

Proof. Let $n \in \mathbb{N}$. Set $p = T(n)$. Suppose for some $k \in \mathbb{N}$, $F_k = T(n) = F_{i_1}$. By Theorem 2, $F_{i_2} \geq F_{k+2}$. Then, by Lemma 10, $F_{i_2} \geq F_{k+2} > 2F_k = 2p$. By uniqueness of the ZRT, $n - p = F_{i_r} + \cdots + F_{i_2}$ and $(n - p, 2p)$ has $T(n - p) = F_{i_2} > 2p$. Hence, $(n - p, 2p)$ is a losing position. □

For now, we state that not every tail may always be taken from n to produce a losing position. In the following subsections, we will prove this rigorously and derive results which show exactly which tails may be removed to put our opponent in a losing position. Theorem 13 is this section's main result. Namely, it proves that removing length j tails of n are the *only* winning moves for $n \in \mathbb{N}$.

Theorem 13 (Fundamental theorem of Fibonacci Nim). *Let $n \in \mathbb{N}$. Then, for any $p \notin \{T_{r-1}(n), T_{r-2}(n), \ldots, T(n)\}$, $(n-p, 2p)$ is a winning position.*

Proof. Let $n \in \mathbb{N}$ and suppose our opponent has removed p tokens. Then the current position is $(n-p, 2p)$. Assume $T(n-p) > 2p$, that is, $(n-p, 2p)$ is a losing position. If $p = T_j(n)$ for some $r > j \geq 1$, then $p \in \{T_{r-1}(n), T_{r-2}(n), \ldots, T(n)\}$. This leaves two cases to examine: (1) p is a sum of terms F_{i_t} where $r \geq t \geq 1$ and $p \neq T_j(n)$ for some $r > j \geq 1$ or (2) $p \neq T_j(n)$ for some $r > j \geq 1$ and p is not of the form given in case (1).

Case 1: Our opponent removes $p = a_r F_{i_r} + a_{r-1} F_{i_{r-1}} + \cdots + a_1 F_{i_1}$ where each $a_j \in \{0, 1\}$ for $j \in [1, i_r]$ and there exists at least one pair (a_j, a_{j+1}) such that $a_j = 0$ and $a_{j+1} = 1$ in the representation of p. Then, $p \neq T_j(n)$ for some $r > j \geq 1$. Without loss of generality, let (a_j, a_{j+1}) be the minimal pair such that $a_j = 0$ and $a_{j+1} = 1$ in the representation of p. Define

$$n' = (F_{i_r} - a_r F_{i_r}) + \cdots + (F_{i_{j+1}} - a_{j+1} F_{i_{j+1}}) + (F_{i_{j-1}} - a_{j-1} F_{i_{j-1}}) + \cdots + (F_{i_1} - a_1 F_{i_1}).$$

Then,

$$n - p = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1} - (a_r F_{i_r} + a_{r-1} F_{i_{r-1}} + \cdots + a_1 F_{i_1}) = n' + F_{i_j}$$

which is a valid Zeckendorff representation and hence the only representation of $n-p$. Since (a_j, a_{j+1}) is minimal, $T(n-p) = F_{i_j}$. We have $2p > F_{i_{j+1}} > T(n-p)$, thus $(n-p, 2p)$ is a winning position and we have reached a contradiction.

Case 2: Our opponent removes p tokens such that $p \neq a_r F_{i_r} + a_{r-1} F_{i_{r-1}} + \cdots + a_1 F_{i_1}$ where each $a_j \in \{0, 1\}$. Since $(n-p, 2p)$ is a losing position, by Lemma 11 we must have $p > T(n)$. Without loss, let $T_j(n)$ for $r > j \geq 1$ be the minimal tail such that $p > T_j(n)$. By assumption, $p \neq T_j(n)$. We have $F_{i_{j+1}} + T_j(n) > p > T_j(n)$ so that $F_{i_{j+1}} > p - T_j(n) > 0$. Define $\delta p = p - T_j(n)$ so that $p = T_j(n) + \delta p$. Let $m = n - T_j(n)$. Then, $n - p = m + T_j(n) - (T_j(n) + \delta p) = m - \delta p$. Since $T(m) > \delta p$, by Lemma 11 and the uniqueness of Zeckendorff representations, $(m - \delta p, 2\delta p)$ is a winning position. It follows that $2p > 2\delta p \geq T(m - \delta p) = T(n-p)$. Therefore, $(n-p, 2p)$ is a winning position and we have reached a contradiction.

Hence, removing some $p \neq T_j(n)$ for some $r > j \geq 1$ results in a winning position. Since there is only one other possible move, removing some tail $F_j(n)$, it follows that if $(n-p, 2p)$ is a losing move, then $p = T_j(n)$ for some $r > j \geq 1$. □

Remark 14. By Definition 9 and Theorem 13, removing $T_j(n)$ tokens where $r > j \geq 1$ will force an immediate losing position to our opponent when $F_{i_{j+1}} > 2T_j(n)$.

In section (2) we have shown that the only possible winning moves in Fibonacci Nim are those that are partial consecutive* sums or, tails of the Zeckendorff representation of the number of tokens in that turn. In the next section, we determine which tails force losing positions and how to identify these tails based solely on the Zeckendorff representation for a given n.

3. Winning tails

In this Section, we will show how to take the result from Remark 14: *removing* $T_j(n)$ *tokens where* $r > j \geq 1$ *will force an immediate losing position to our opponent when* $F_{i_{j+1}} > 2T_j(n)$ and identify which tails satisfy this condition. Existence of winning moves was proved for *Dynamic One-Pile Nim* in a paper by Holshouser, Reiter and Rudzinski [2003]; Fibonacci Nim is classified as a dynamic one-pile Nim game in their paper. Below, we validate the existence of these moves as well as carefully show exactly how to find these winning moves. In addition, we have included a table at the end of this paper to present these results for the first 90 positive integers.

We are concerned with which tails can be taken and which cannot. That is, if $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$, when is $F_{i_{j+1}} > 2T_j(n)$ for $r > j \geq 1$? We accomplish this by looking at an arbitrary tail $T_j(n)$ of n. We classify exactly when taking $T_j(n)$ results in leaving a losing position to our opponent.

We begin by setting $a_{j+1} = i_{j+1} - i_j$ and $a_j = i_j - i_{j-1}$. Then, a_{j+1} and a_j are the differences in the subscripts of consecutive* Fibonacci numbers in a Zeckendorff representation of n. In this section we will show that for any F_{i_j}, by considering the "gaps" around it, where the gaps are the differences above, we can determine if removing $T_{i_j}(n)$ tokens give our opponent a losing position. For us to do this, we must first introduce the *gap-vector*.

Definition 15. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$. We define the *gap-vector* of n to be $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$ where $a_r = i_r - i_{r-1}$, $a_{r-1} = i_{r-1} - i_{r-2}$, $\ldots$, $a_2 = i_2 - i_1$, and $a_1 = i_1$. We also define $|G(n)| = r$, where r is the number of summands in the Zeckendorff representation of n.

Example 16. Let $n = 129 = F_{11} + F_9 + F_5 + F_2$. Then,

$$G(129) = (11-9, 9-5, 5-2; 2) = (2, 4, 3; 2) \quad \text{and} \quad |G(129)| = 4.$$

The gap-vector of n shows the difference of the subscripts of the consecutive* Fibonacci numbers in the Zeckendorff representation of n (again, consecutive* refers to the subscripts i_j, i_{j+1} for some $j \in \mathbb{N}$). The last coordinate of the gap-vector is the subscript of the smallest Fibonacci number present in the Zeckendorff representation of n. It follows that we can reconstruct n by using

Example 17. Let $G(n) = (2, 4, 3; 2)$. Then, F_2 is the first Fibonacci number in the representation of n. From here, we can build the rest of the numbers: 2+3 = 5, so F_5 is the next number; 5+4 = 9, so F_9 is the third number; and 9+2=11, so F_{11} is the last number in the representation of n. Hence, $n = F_{11} + F_9 + F_5 + F_2 = 129$.

It is worth mentioning that by the ZRT each $a_j \geq 2$ for $j \in \mathbb{N}$. We now begin to examine which tails provide winning moves. Consider $p = T_j(n)$ for some $n, j \in \mathbb{N}$. We will classify exactly when $T_j(n)$ is a winning move and hence leaves the opponent the losing position $(n - p, 2p)$.

Notational remark. For the following lemmas, we introduce the symbol $(k : 2)$ such that $(k : 2) \in \{2, 3\}$ where $(k : 2) \equiv k \bmod 2$. Similarly, $(k : 3) \in \{2, 3, 4\}$ where $(k : 3) \equiv k \bmod 3$. For example, $F_8 + \cdots + F_{8:2} = F_8 + \cdots + F_2$ since $(8 : 2) \equiv 8 \bmod 2$ and $(8 : 2) \in \{2, 3\}$.

For the remainder of this section, we will give a lemma and then a corollary. The lemma provides properties of particular Fibonacci series. The corollaries tie the lemma into Fibonacci Nim.

Lemma 18. *For $k \geq 5$, we have $F_k > 2(F_{k-3} + F_{k-5} + \cdots + F_{k:2})$.*

Proof. For $k = 5$ and $k = 6$,

$$F_5 = 5 > 2(1) = 2(F_2),$$
$$F_6 = 8 > 4 = 2(2) = 2(F_3).$$

Suppose $F_k > 2(F_{k-3} + F_{k-5} + \cdots + F_{k:2})$. Then by the induction hypothesis, $2F_{k-1} + F_k > 2F_{k-1} + 2(F_{k-3} + F_{k-5} + \cdots + F_{k:2}) = 2(F_{k-1} + \cdots + F_{k:2})$. But, $F_{k+2} = F_{k+1} + F_k > 2F_{k-1} + F_k$ by Lemma 10. Hence, $F_{k+2} > 2(F_{k-1} + \cdots + F_{k:2})$. □

Corollary 19. *Let $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. If $a_{q+1} \geq 3$ for some $r > q > 1$, then $(n - p, 2p)$ is a losing position for $p = T_q(n)$.*

Proof. Let $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. Suppose $a_{q+1} \geq 3$ for some $r > q > 1$ and set $p = T_q(n)$. Then $i_{q+1} \geq i_q + 3$. By Lemma 18, we have $F_{i_{q+1}} > 2(F_{i_q} + \cdots + F_{i_1}) = 2T_q(n)$. We have, $n - p = F_{i_r} + \cdots + F_{i_{q+1}}$ by the uniqueness of Zeckendorff representations, hence $T(n - p) = F_{i_{q+1}} > 2T_q(n) = 2p$ and $(n - p, 2p)$ is a losing position. □

We see by the above corollary that if $G(n) = (a_r, \ldots, a_2; a_1)$ contains coordinates $a_j \geq 2$ and some $a_{q+1} \geq 3$ we can always remove the tail beginning with the Fibonacci number F_{i_q}. But notice, by the ZRT, every representation will have $a_j \geq 2$ for $r \geq j \geq 2$. Hence, we have just shown by Corollary 19 that given some $n = F_{i_r} + \cdots + F_{i_{j+1}} + F_{i_j} + \cdots + F_{i_1}$, if $i_{j+1} - 3 \geq i_j$, then removing $p = T_j(n)$ results in $(n - p, 2p)$ being a *losing position*. Therefore it follows that we need only to consider when $i_{j+1} - 2 = i_j$ to classify the remainder of winning tails.

Lemma 20. *For $k \geq 8$, we have $F_k > 2(F_{k-2} + F_{k-6} + F_{k-8} + \cdots + F_{k:2})$.*

Proof. For $k = 8$ and $k = 9$,

$$F_8 = 21 > 2(8+1) = 2(F_6 + F_2),$$
$$F_9 = 34 > 30 = 2(13+2) = 2(F_7 + F_3).$$

Assume

$$F_k > 2(F_{k-2} + F_{k-6} + F_{k-8} + \cdots + F_{k:2}).$$

By the induction hypothesis we have,

$$F_{k+2} = F_{k+1} + F_k > F_{k+1} + 2F_{k-2} + 2(F_{k-6} + \cdots + F_{k:2}).$$

But, $F_{k+1} + 2F_{k-2} = F_k + F_{k-1} + 2F_{k-3} + 2F_{k-4}$. By Lemma 10, $2F_{k-3} > F_{k-2}$. Hence, $F_{k+1} + 2F_{k-2} > F_k + F_{k-1} + F_{k-2} + 2F_{k-4} = 2(F_k + F_{k-4})$. Therefore, $F_{k+2} > 2(F_k + F_{k-4} + F_{k-6} + \cdots + F_{k:2})$. □

Corollary 21. *Let $G(n) = (a_r, a_{r-1}, \dots, a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. If $a_q \geq 4$ and $a_{q+1} = 2$ for some $r \geq q > 1$, then $(n-p, 2p)$ is a losing position for $p = T_q(n)$.*

Proof. Let $G(n) = (a_r, a_{r-1}, \dots, a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. Suppose that $a_q \geq 4$ for some $r \geq q > 1$ and set $p = T_q(n)$. Then, $i_{q+1} - 2 = i_q \geq i_{q-1} + 4$. By Lemma 20, we have $F_{i_{q+1}} > 2(F_{i_q} + \cdots + F_{i_1}) = 2T_q(n)$. We have, $n - p = F_{i_r} + \cdots + F_{i_{q+1}}$ by the uniqueness of Zeckendorff representations, hence $T(n-p) = F_{i_{q+1}} > 2T_q(n) = 2p$ and $(n-p, 2p)$ is a losing position. □

We see by Corollary 21 that if $G(n) = (a_r, \dots, a_2; a_1)$ contains coordinates $a_j \geq 2$ and some $a_q \geq 4$ and $a_{q+1} = 2$, we can always remove the tail beginning with the Fibonacci number F_{i_q}. Hence, we have just shown that given some $n = F_{i_r} + \cdots + F_{i_{j+1}} + F_{i_j} + \cdots + F_{i_1}$, if $i_{q+1} - 2 = i_q \geq i_{q-1} + 4$, then removing $p = T_j(n)$ results in $(n-p, 2p)$ being a *losing position*. By Corollaries 19 and 21, we have just shown that if we have $a_{q+1} \geq 3$ or, if $a_{q+1} = 2$ and $a_q \geq 4$, then $p = T_q(n)$ is a winning move, that is, $(n-p, 2p)$ is a losing position. Thus, what remains to examine are the cases $a_{q+1} = 2 = a_q$ and $a_{q+1} = 2$ and $a_q = 3$. We begin with the former.

Lemma 22. *For $k \geq 6$, we have $F_k \leq 2(F_{k-2} + F_{k-4})$.*

Proof. Let $k = 6$. Then, $F_6 = 8 = 2(3+1) = 2(F_4 + F_2)$. For any $k > 6$, we have $F_k = 2F_{k-2} + F_{k-3}$. By Lemma 11, $2F_{k-4} \geq F_{k-3}$. Hence, $F_k = 2F_{k-2} + F_{k-3} \leq 2(F_{k-2} + F_{k-4})$. □

Corollary 23. *Let $G(n) = (a_r, a_{r-1}, \dots, a_2; a_1)$ where $r > 2$ and $a_j \geq 2$ for $r \geq j > 1$. If $a_{q+1} = 2 = a_q$ for some $r \geq q > 1$, then $(n-p, 2p)$ is a winning position for $p = T_q(n)$.*

Proof. Let $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. Suppose $a_{q+1} = 2 = a_q$ for some $r \geq q > 1$ and set $p = T_q(n)$. Then $i_{q+1} - 2 = i_q = i_{q-1} + 2$. By Lemma 22, we have $F_{i_{q+1}} \leq 2(F_{i_q} + F_{i_{q-1}}) \leq 2(F_{i_q} + \cdots + F_{i_1}) = 2T_q(n)$. We have, $n - p = F_{i_r} + \cdots + F_{i_{q+1}}$ by the uniqueness of Zeckendorff representations, but $T(n-p) = F_{i_{q+1}} \leq 2T_q(n) = 2p$. Thus, $(n-p, 2p)$ is a winning position. □

We are now left with the case $a_{q+1} = 2$ and $a_q = 3$. It turns out, this case is slightly more complicated than the previous cases. We will show that given $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$, if there exists some q such that every $a_k \geq 3$ for $r > q \geq k > 1$, then $T_q(n)$ for $r > q > 1$ is a winning move. If however, we have some $a_k = 2$ for $q \geq k > 1$, then $T_q(n)$ for $r > q > 1$ is a losing move. We begin with the former.

Lemma 24. *For* $k \geq 10$, $F_k - 2(F_{k-2} + F_{k-5} + F_{k-8} + \cdots + F_{k:3}) = q$ *where* $q \in \{1, 2\}$.

Proof. We prove the lemma in cases for $F_{k:3}$. Specifically for some $m \in \mathbb{N}$ and $m \geq 3$, $F_{k:3} = F_2$ when $k = 3m+1$ since $3m+1-(2+3(m-1)) = 2$ and $F_{k:3} = F_3$ when $k = 3m+2$ since $3m+2-(2+3(m-1)) = 3$. $F_{k:3} = F_4$ when $k = 3m$ since $3m - (2+3(m-2)) = 4$. Note, if we have $3m - (2+3(m-1)) = 1$, we will not have a valid Zeckendorff representation, hence we must reduce our multiple by one, which yields $3(m-2)$ above.

Case 1. Let $F_{k:3} = F_2$ and let $m = 3$ so that $k = 3m+1 = 10$. In this case, $F_{10} - 2(F_8 + F_5 + F_2) = 55 - 2(21 + 5 + 1) = 1$. Let $m > 3$ so that $k > 10$ and assume that $F_{3m+1} - 2(F_{3m-1} + F_{3m-4} + F_{3m-7} + \cdots + F_5 + F_2) = 1$. Then, $F_{3m+1} + 2F_{3m+2} - 2F_{3m+2} - 2(F_{3m-1} + F_{3m-4} + \cdots + F_5 + F_2) = 1$ by the inductive hypothesis. But, $F_{3(m+1)+1} = F_{3m+4} = F_{3m+3} + F_{3m+2} = 2F_{3m+2} + F_{3m+1}$. Hence, $F_{3m+4} - 2(F_{3m+2} + F_{3m-1} + F_{3m-4} + \cdots + F_5 + F_2) = 1$.

Case 2. Now suppose that $F_{k:3} = F_3$ and let $m = 3$ so that $k = 11$. In this case, $F_{11} - 2(F_9 + F_6 + F_3) = 89 - 2(34 + 8 + 2) = 1$. Let $m > 3$ so that $k > 11$ and assume that $F_{3m+2} - 2(F_{3m} + F_{3m-3} + F_{3m-6} + \cdots + F_6 + F_3) = 1$. Then $F_{3m+2} + 2F_{3m+3} - 2F_{3m+3} - 2(F_{3m} + F_{3m-3} + \cdots + F_6 + F_3) = 1$ by the inductive hypothesis. But, $F_{3(m+1)+2} = F_{3m+5} = F_{3m+4} + F_{3m+3} = 2F_{3m+3} + F_{3m+2}$. Hence, $F_{3m+5} - 2(F_{3m+3} + F_{3m} + F_{3m-3} + \cdots + F_6 + F_3) = 1$.

Case 3. Finally, let $F_{k:3} = F_4$ and let $m = 4$ so that $k = 12$. Here we have $F_{12} - 2(F_{10} + F_7 + F_4) = 144 - 2(55 + 13 + 3) = 2$. Let $m > 4$ so that $k > 12$ and assume that $F_{3m} - 2(F_{3m-2} + F_{3m-5} + F_{3m-8} + \cdots + F_7 + F_4) = 2$. Then, $F_{3m} + 2F_{3m+1} - 2F_{3m+1} - 2(F_{3m-2} + F_{3m-5} + \cdots + F_7 + F_4) = 2$ by the inductive hypothesis. But, $F_{3(m+1)} = F_{3m+3} = F_{3m+2} + F_{3m+1} = 2F_{3m+1} + F_{3m}$. So, $F_{3m+3} - 2(F_{3m+1} + F_{3m-2} + F_{3m-5} + \cdots + F_7 + F_4) = 2$.

Hence, in each case we find with $q \in \{1, 2\}$ that

$$F_k - 2(F_{k-2} + F_{k-5} + F_{k-8} + \cdots + F_{k:3}) = q. \qquad \square$$

Remark 25. It should be clear from Lemma 24 that for $k \geq 10$,

$$F_k > 2((F_{k-2} + F_{k-5} + F_{k-8} + \cdots + F_{k:3}).$$

Corollary 26. *Let $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$ where $r > 2$ and $a_j \geq 2$ for $r \geq j > 1$. If $a_{q+1} = 2$ and $a_j \geq 3$ for $q \geq j \geq 1$, then $(n - p, 2p)$ is a losing position for $p = T_q(n)$.*

Proof. Let $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$ where $r > 2$ and $a_j \geq 2$ for $r \geq j > 1$. Suppose $a_{q+1} = 2$ and $a_j \geq 3$ for $q \geq j \geq 1$ and set $p = T_q(n)$. Then $i_{q+1} - 2 = i_q$ and $i_{j+1} - 3 \geq i_j$ for every $q > j \geq 1$. By Lemma 24 and Remark 25, we have $F_{i_{q+1}} > 2(F_{i_q} + \cdots + F_{i_1}) = 2T_q(n)$. We have, $n - p = F_{i_r} + \cdots + F_{i_{q+1}}$ by the uniqueness of Zeckendorff representations and $T(n - p) = F_{i_{q+1}} > 2T_q(n) = 2p$. Thus, $(n - p, 2p)$ is a losing position. $\square$

By Corollary 26, if $G(n) = (a_r, \ldots, a_2; a_1)$ contains coordinates $a_j \geq 2$ and if for some $a_{q+1} = 2$ we have for every $q \geq k \geq 1$, $a_k \geq 3$ then we may remove the tail beginning with the Fibonacci number F_{i_q}, that is, $T_q(n)$. All that remains to show is the case when at least one $a_k = 2$.

Lemma 27. *For $k \geq 6$, we have $F_k - (F_{k-1} + F_{k-4} + F_{k-7} + \cdots + F_{k:3}) > 1$.*

Proof. We prove the lemma in cases for $F_{k:3}$. Specifically for some $m \in \mathbb{N}$ and $m \geq 2$, there are three distinct possibilities: either $F_{k:3} = F_2$ when $k = 3m$ since $3m-(1+3(m-1)) = 2$ or $F_{k:3} = F_3$ when $k = 3m+1$ since $3m+2-(1+3(m-1)) = 3$ or $F_{k:3} = F_4$ when $k = 3m + 2$ since $3m + 2 - (1 + 3(m - 1)) = 4$.

Case 1. Let $m = 2$ so that $k = 6$. Then, $F_6 - (F_5 + F_2) = 8 - (5 + 1) = 2$. Assume $F_k - (F_{k-1} + F_{k-4} + F_{k-7} + \cdots + F_{k:3}) > 1$ for $m > 2$. Then, by induction hypothesis, we have $F_{3m} + F_{3m+2} - F_{3m+2} - (F_{3m-1} + F_{3m-4} + \cdots + F_2) > 1$. But, $F_{3m+3} = F_{3m+2} + F_{3m+1} > F_{3m+2} + F_{3m}$ and $F_{3m+1} - F_{3m} > 2$ when $m > 2$ by construction. Hence, $F_{3m+3} - (F_{3m+2} + F_{3m-1} + \cdots + F_2) > 1$.

In Case 2, we replace $k = 3m$ with $k = 3m + 1$ and in Case 3 we replace $k = 3m$ with $k = 3m + 2$. The arguments are then the same as that of Case 1. $\square$

Corollary 28. *Let $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$. If every $a_j = 3$ for some $r > j > 1$ but there exists at least one $a_q = 2$ such that $j > q \geq 1$, then for $p = T_j(n)$, $(n-p, 2p)$ is a winning position.*

Proof. Let $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$. Suppose that every $a_j = 3$ for some $r > j > 1$ except for some $a_q = 2$ such that $j > q \geq 1$ and set $p = T_j(n)$. Define $G(n') = (b_r, b_{r-1}, \ldots, r_2; r_1)$ where each $b_j = 3$ for $r \geq j > 1$ and $b_1 = a_1$. Then, by definitions 6 and 15, if $T_q(n) = F_{i_q} + F_{i_{q-1}} + \cdots + F_{i_1}$ then $T_q(n) = F_{i_q-1} +$

$F_{i_{q-1}-1} + \cdots + F_{i_1-1}$. If $i_1 - 1 = 1$, then $T_q(n')$ terminates with F_{i_2-1}, which will make no difference in the following argument. By Lemma 24, $F_{i_{q+1}} - 2T_q(n') = g$ where $g \in \{1, 2\}$. By Lemma 27, $F_{i_q+1} \geq T_q(n') + 2$. Therefore,

$$F_{i_{q+1}} - 2T_q(n) \leq F_{i_{q+1}} - 2(T_q(n') + 2) = g - 4.$$

Since $g \in \{1, 2\}$, $g - 4 < 0$. This immediately shows that

$$T(n - p) = F_{i_{q+1}} \leq 2T_q(n) = 2p$$

and hence $(n - p, 2p)$ is a winning position. □

We have now fully characterized when $T_j(n)$ is a winning move based solely on the *gap-vectors* of n. We present a table below to summarize this section's findings. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots F_{i_1}$. Then, $G(n) = (a_r, a_{r-1}, \ldots, a_2; a_1)$. Recall, each $a_j \geq 2$ by construction. Let the tail in question be $T_j(n)$. Then the "gaps" that surround F_{i_j} are precisely a_{j+1} and a_j. We have the following:

a_{j+1}	a_j	Further Conditions	Winning Move
≥ 3	≥ 2	None	Yes
2	≥ 4	None	Yes
2	2	None	No
2	3	$j \geq q \geq 1,\ a_q \geq 3$	Yes
2	3	$\exists q$ for $j \geq q \geq 1,\ a_q = 2$	No

Thus, by knowing the Zeckendorff representation of n, we may now find all possible winning moves, or moves that make $(n - p, 2p)$ a losing position.

In Table 1, we present these results for $n \in [1, 90] \subset \mathbb{N}$. First, recall the first 11 Fibonacci numbers: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$, $F_{11} = 89$. The column 'Zeck.' gives the Zeckendorf representation in vector form, where the rightmost number is the coefficient of F_2, for example, $17 = F_7 + F_4 + F_2 = (100101)$. The last column lists the sum of each winning tail. Continuing with $n = 17$, we have $G(17) = (3, 2; 2)$ and by the table above, we see that taking $F_2 = 1$ and $F_4 + F_2 = 3 + 1 = 4$ are both winning moves.

4. Skilled vs unskilled players and probabilities of an unskilled win

We begin this section by noting that in order for an unskilled player to win against a skilled player, (1) the unskilled player must go first and always make a winning move, or, (2) the skilled player must start from $n = F_k$ for some $n, k \in \mathbb{N}$. If not, the skilled player will immediately gain control of the game and provided the skilled player doesn't make any mistakes, he will force a win over the nonskilled player. It is from this perspective that we discuss probabilities of an unskilled win. For

n	Zeck.	Moves	n	Zeck.	Moves	n	Zeck.	Moves
1	(1)	1	31	(1010010)	2; 10	61	(100001001)	1; 6
2	(10)	2	32	(1010100)	3	62	(100001010)	2; 7
3	(100)	3	33	(1010101)	1	63	(100010000)	8
4	(101)	1	34	(10000000)	34	64	(100010001)	1; 9
5	(1000)	5	35	(10000001)	1	65	(100010010)	2; 10
6	(1001)	1	36	(10000010)	2	66	(100010100)	3; 11
7	(1010)	2	37	(10000100)	3	67	(100010101)	1; 12
8	(10000)	8	38	(10000101)	1; 4	68	(100100000)	13
9	(10001)	1	39	(10001000)	5	69	(100100001)	1; 14
10	(10010)	2	40	(10001001)	1; 6	70	(100100010)	2; 15
11	(10100)	3	41	(10001010)	2; 7	71	(100100100)	3; 16
12	(10101)	1	42	(10010000)	8	72	(100100101)	1; 4; 17
13	(100000)	13	43	(10010001)	1; 9	73	(100101000)	5; 18
14	(100001)	1	44	(10010010)	2; 10	74	(100101001)	1; 6; 19
15	(100010)	2	45	(10010100)	3; 11	75	(100101010)	2; 20
16	(100100)	3	46	(10010101)	1; 12	76	(101000000)	21
17	(100101)	1; 4	47	(10100000)	13	77	(101000001)	1; 22
18	(101000)	5	48	(10100001)	1; 14	78	(101000010)	2; 23
19	(101001)	1; 6	49	(10100010)	2; 15	79	(101000100)	3; 24
20	(101010)	2	50	(10100100)	3; 16	80	(101000101)	1; 4; 25
21	(1000000)	21	51	(10100101)	1; 4	81	(101001000)	5; 26
22	(1000001)	1	52	(10101000)	5	82	(101001001)	1; 6; 27
23	(1000010)	2	53	(10101001)	1; 6	83	(101001010)	2; 7
24	(1000100)	3	54	(10101010)	2	84	(101010000)	8
25	(1000101)	1; 4	55	(100000000)	55	85	(101010001)	1; 9
26	(1001000)	5	56	(100000001)	1	86	(101010010)	2; 10
27	(1001001)	5; 6	57	(100000010)	2	87	(101010100)	3
28	(1001010)	2; 7	58	(100000100)	3	88	(101010101)	1
29	(1010000)	8	59	(100000101)	1; 4	89	(1000000000)	89
30	(1010001)	1; 9	60	(100001000)	5	90	(1000000001)	1

Table 1. Zeckendorff representations and winning tail sums.

the remainder of this section, we assume that the unskilled player removes tokens randomly and that the skilled player is free from making errors. Further, we commit to the following strategy for a skilled player in a losing position:

Losing position strategy (LPS). *If the skilled player is currently playing from a losing position, then he removes one token.*

Therefore, by Definition 9 if the skilled player is given a position $(n, 2p')$ such that $T(n) > 2p'$, then set $p = 1$ and give the opponent the position $(n-1, 2)$. Hence, the unskilled player may take either one or two tokens on their next turn.

Lemma 29. *Let the current position be* $(n, 2)$ *to the unskilled player. Then for* $p \in \{1, 2\}$, *we have* $P\big[(n-p, 2p) = \text{losing position}\big] \le \frac{1}{2}$.

Proof. Assume the unskilled player has position $(n, 2)$ where

$$n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}.$$

If $F_{i_1} = 1 = F_2$, then $p = 1$ leaves $n-1 = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_2}$ but $T(n-1) \ge F_4 = 3 > 2(1) = 2p$. Now, $p = 2$ leaves $(n-2, 4)$. Since $2 = p \ne T_j(n)$, by Theorem 13, $(n-2, 4)$ is a winning position. Now suppose $F_{i_1} = 2 = F_3$, then the role of $p = 1$ and $p = 2$ are the reverse of case 1. Finally, If $F_{i_1} = m \ge 3$, then $T(n) = F_{i_1} > 2$ by the ZRT. Then, by Lemma 11, $(n-p, 2p)$ where $p = 1$ or $p = 2$ is a winning position. Hence, in all three instances, $P\big[(n-p, 2p) = \text{losing position}\big] \le \frac{1}{2}$. ☐

Lemma 30. *Let* $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$. *Then,* $\big|G(n)\big| \le \left\lfloor \frac{i_r}{2} \right\rfloor$.

Proof. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ and suppose $F_{i_r} = F_k$ from some k. Define $n' = F_{i_{r'}} + F_{i_{r-1'}} + \cdots + F_{i_{1'}}$ such that $F_{i_{r'}} = F_k$ and $G(n') = (2, 2, \ldots, 2; 2)$. Let $k = 2m$ for some $m \in \mathbb{N}$. Recall, every $a_j \ge 2$ by the ZRT. Since there are $(2m-2)/2 + 1 = m$ multiples of $2 \in [2, k]$, we have $m = k/2 = |G(n')|$. Suppose $r > m$. Then by Corollary 4 and Definition 15, $r = |G(n)| > m$ implies that $F_{i_r} > F_k$ which is a contradiction. Now let $k = 2m + 1$. Note that $\lfloor k/2 \rfloor = m$. Let n' be defined such that $G(n') = (a_{r'}, a_{r-1'}, \ldots, a_{2'}; a_{1'})$ where $F_{i_{r'}} = F_k$ and each $a_{j'} = 2$ except for some $a_{k'} = 3$ where $r' \ge k' \ge 1'$. Since there are

$$\left\lfloor \frac{(2m+1)-2}{2} + 1 \right\rfloor = m$$

multiples of $2 \in [2, k]$, we have $m = \lfloor k/2 \rfloor = |G(n')|$. Suppose $r > m$. Then by Corollary 4 and Definition 15, $r = |G(n)| > m$ implies that $F_{i_r} > F_k$ which is a contradiction. ☐

Lemma 30 gives an upper bound on the number of terms in the Zeckendorff representation of some n.

Lemma 31. *For* $k \ge 5$, $F_k \ge \dfrac{p^k - 0.1}{\sqrt{5}}$, *where* $p = \dfrac{\sqrt{5}+1}{2}$.

Proof. The closed form of Fibonacci numbers is given by,

$$F_k = \frac{p^k - (-p)^{-k}}{\sqrt{5}}, \text{ where } p = \frac{\sqrt{5}+1}{2}$$

(see, e.g., [Bóna 2002]). Then, we have

$$\begin{aligned}(-p)^{-5} &\approx -0.09016994\\(-p)^{-6} &\approx \quad 0.05572809.\end{aligned}$$

By simple application of the derivative test from elementary calculus, we see that this is a decreasing function for all $k \geq 5$. Hence, we have that $-0.1 \leq (-p)^{-k} \leq 0.1$ for all $k \geq 5$. Then for $k \geq 5$, we have

$$F_k \geq \frac{p^k - 0.1}{\sqrt{5}}. \qquad \square$$

Corollary 32. *Let the current position be* $(n, n-1)$ *to the unskilled player where* $n \geq 5$ *and* $F_{k+1} > n \geq F_k$, *then*

$$P\big[p = T_j(n)\big] \leq \frac{k\sqrt{5}}{2(p^k - 0.1)}$$

where $1 \leq j \leq k$ *and* p *is the unskilled player's next move.*

Proof. If $n = F_k$, then $P[p = T_j(n)] = 0$ since the only tail is $F_k = n$ and the unskilled player may remove at most $n-1$ tokens. Let $F_{k+1} > n > F_k$ so that the number of terms in the Zeckendorf representation of n is at most $\frac{k}{2}$ terms by Lemma 30 and hence at most $\frac{k}{2}$ possible tails. Then, since there are at least F_k possible choices for p, by Lemma 31 we have for $1 \leq j \leq k$,

$$P[p = T_j(n)] = \frac{k/2}{(p^k - 0.1)/\sqrt{5}} = \frac{k\sqrt{5}}{2(p^k - 0.1)}. \qquad \square$$

Corollary 32 shows that if an unskilled player begins the game where $n \geq 5$, then the probability that the unskilled player chooses p such that p is a winning move is less than $\frac{2}{5}$ and by elementary calculus, the probability function $P[p = T_j(n)]$ can be shown to be a decreasing function for $k \geq 5$ so that as n increases, the probability that an unskilled player will choose a winning move from the beginning position (or any other of the form $(n, n-1)$) decreases exponentially. Note, if $n = 3$ then the first player will lose and if $n = 4$, then only winning move the first player may take is $p = 1$, thus the first player has a probability of $\frac{1}{4} < \frac{2}{5}$ of correctly choosing a tail.

We now have everything in place to state the main result of this section. This upper bound is dependent on the first move of the unskilled player however, and therefore cannot be calculated explicitly before the game begins.

Theorem 33. *Let* $n > p$ *and* $(n-p, m)$ *be the first position to the skilled player where* $m \in \{n-1, 2p\}$. *Set* $n' = n - p$. *Then, using the* LPS,

(1) *if* $n \neq F_k$ *for some* $k \geq 4$ *then*

$$P\big[\text{Unskilled player wins}\big] \leq \frac{1}{5(2^{b-1})}, \quad \textit{where} \quad b = \left\lfloor \frac{n'}{3} \right\rfloor;$$

(2) *if* $n = F_k$ *for some* $k \geq 5$, *then*

$$P\big[\text{Unskilled player wins}\big] \leq \frac{1}{2^b}, \quad \textit{where} \quad b = \left\lfloor \frac{n'}{3} \right\rfloor.$$

Proof. There are two nontrivial cases needed to prove the result.

Case 1. $n \neq F_k$ for some $n, k \in \mathbb{N}$. If the skilled player starts, he wins every time. Thus, skilled player receives the position $(n-p, 2p)$ where $n-1 \geq p \geq 1$. By LPS, after the initial turn, the unskilled player will always receive $(k, 2)$ for some $k < n$ and by Lemma 29, $P[(n-p', 2p') = \text{losing position}] \leq \frac{1}{2}$ where $p' \in \{1, 2\}$. Hence, at most, 3 tokens are removed after one round of play. Let $n' = n - p$, then there will be at least $\lfloor n'/3 \rfloor$ rounds played from this point in the game. By Corollary 32 and repeated use of Lemma 29, we find that

$$P\big[\text{Unskilled player wins}\big] \leq \left(\frac{2}{5}\right)\left(\frac{1}{2^{\lfloor n'/3 \rfloor}}\right) = \frac{1}{5(2^{b-1})}, \quad \text{where} \quad b = \left\lfloor \frac{n'}{3} \right\rfloor.$$

Case 2. $n = F_k$ for some $n, k \in \mathbb{N}$. By Lemma 11, removing p tokens make $(n-p, 2p)$ a winning position. Hence, the unskilled player loses if he goes first. Now assume the skilled player begins and by LPS, takes $1 < T(n)$ tokens. By Lemma 11, $(n-1, 2)$ is a winning position. Thus, this position is that of Case 1, where the unskilled player doesn't have the free move: $(n, n-1)$. Hence,

$$P\big[\text{Unskilled player wins}\big] \leq \frac{1}{2^{\lfloor n'/3 \rfloor}} = \frac{1}{2^b}, \quad \text{where} \quad b = \left\lfloor \frac{n'}{3} \right\rfloor. \qquad \square$$

5. Final remarks

In this paper we have characterized all winning algorithms for the game Fibonacci Nim. We have shown that the known winning algorithm is just a particular case of the generalized wining algorithm. In addition, we have shown an upper bound on the probability that an unskilled player may beat a skilled player if our unskilled player guesses randomly and our skilled player plays according to our losing position strategy.

Future research may look into different losing position strategies as well as different types of unskilled players. For example, as a second losing position strategy, by taking more than one token from a losing position, we may find a tighter upper bound on the probability that the unskilled player wins. Additionally, we could introduce a semiskilled player, one whose guesses are not random but are based on some rule.

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