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The nonexistence of cubic Legendre multiplier sequences

Tamás Forgács, James Haley, Rebecca Menke and Carlee Simon

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Our main result is the proof of the recently conjectured nonexistence of cubic Legendre multiplier sequences. We also give an alternative proof of the nonexistence of linear Legendre multiplier sequences using a method that will allow for a more methodical treatment of sequences interpolated by higher degree polynomials.

1. Introduction

Given a simple set of polynomials $Q=\{q_k(x)\}_{k=0}^{\infty}$ and a sequence of numbers $\{\gamma_k\}_{k=0}^{\infty}$, one can define the operator associated with $\{\gamma_k\}_{k=0}^{\infty}$ as $T[q_k(x)]=\gamma_k q_k(x)$ for $k=0,1,2,\ldots$, and extend its action to $\mathbb{R}[x]$ linearly. Our work in this paper concerns such operators when Q consists of the Legendre polynomials.

Definition 1. The Legendre polynomials $\mathfrak{Le}_k(x)$ are defined by the generating relation

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{k=0}^{\infty}\mathfrak{Le}_k(x)t^k,$$

where the square root denotes the branch which goes to 1 as $t\to 0$.

Definition 2. A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is a Legendre multiplier sequence if $\sum_{k=0}^{n} a_k\gamma_k\mathfrak{Le}_k(x)$ has only real zeros whenever $\sum_{k=0}^{n} a_k\mathfrak{Le}_k(x)$ has only real zeros. We define Q-multiplier sequences for any basis Q of $\mathbb{R}[x]$ analogously. If Q is the standard basis, the associated multiplier sequences are called classical multiplier sequences (of the first kind).

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Every sequence of the form $0, 0, 0, \dots, a, b, \dots, 0, 0, 0, \dots$, where $a, b \in \mathbb{R}$, is a Legendre multiplier sequence. The literature calls such sequences *trivial*. In addition to these, there is an abundance of *nontrivial* Legendre multiplier sequences (see [Blakeman et al. 2012] for examples). Thus, the problem of characterizing polynomials which interpolate Legendre multiplier sequences is a meaningful one, and it fits well into the landscape of current research in the theory of multiplier sequences (see, for example, [Blakeman et al. 2012; Brändén and Ottergren 2014; Forgács and Piotrowski 2013b; Yoshida 2013]). The present paper contributes to this line of inquiry by settling a conjecture on the nonexistence of cubic Legendre multiplier sequences [Blakeman et al. 2012, Open problem (1)]. In addition, we give a new proof of the nonexistence of linear Legendre multiplier sequences, which is more methodical than the educated hunt for test polynomials whose zeros fail to remain real after having been acted on by a linear sequence.

The rest of the paper is organized as follows. In Section 2 we present a number of known results which are relevant to our investigations. Section 3 exhibits a new proof of the nonexistence of linear Legendre multiplier sequences [Blakeman et al. 2012, Proposition 2] using a theorem of Borcea and Brändén. Our method exploits the fact that one does not need to have full knowledge of all coefficient polynomials $T_k(x)$ of a linear operator $T = \sum_{k=0}^{\infty} T_k(x)D^k$ in order to decide whether or not T is reality preserving. Section 4 contains the main result, Theorem 17, which establishes the nonexistence of cubic Legendre multiplier sequences. We conclude with a section on open problems.

2. Background

Central to the theory of (classical) multiplier sequences is the Laguerre–Pólya class of real entire functions, which we denote by $\mathcal{L}\text{-}\mathcal{P}$. We recall the definition here, along with a recent theorem characterizing this class as precisely those real entire functions which satisfy the generalized Laguerre inequalities.

Definition 3. A real entire function $\varphi(x) = \sum_{k=0}^{\infty}(\gamma_k/k!)x^k$ is said to belong to the Laguerre–Pólya class, written $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, if it can be written in the form

$$\varphi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k},$$

where $b, c \in \mathbb{R}$, $x_k \in \mathbb{R} \setminus \{0\}$, m is a nonnegative integer, $a \geq 0$, $0 \leq \omega \leq \infty$ and $\sum_{k=1}^{\omega} 1/x_k^2 < \infty$. If $\gamma_k \geq 0$ for all $k = 0, 1, 2, \dots$, we say that $\varphi \in \mathcal{L}\text{-}\mathcal{P}^+$.

Csordas and Vishnyakova recently completed the following characterization of the class $\mathcal{L}\text{-}\mathcal{P}$.

Theorem 4 [Csordas and Varga 1990, Theorem 2.9; Csordas and Vishnyakova 2013, Theorem 2.3]. *Let $\varphi(x)$ denote a real entire function, with $\varphi(x) \not\equiv 0$. Then $\varphi \in \mathscr{L}\text{-}\mathscr{P}$ if and only if for all $n \in \mathbb{N}_0$ and for all $x \in \mathbb{R}$,*

$$L_n(x, \varphi) := \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!} \binom{2n}{j} \varphi^{(j)}(x) \varphi^{(2n-j)}(x) \geq 0.$$

We shall make use of this theorem in Section 3 when we reprove the nonexistence of linear Legendre multiplier sequences. Since $\mathscr{L}\text{-}\mathscr{P}$ is exactly the class of real entire functions which are locally uniform limits on $\mathbb{C}$ of real polynomials with only real zeros (see [Levin 1956, Chapter VIII] or [Obreschkoff 1963, Satz 3.2]), it is closed under differentiation. Thus if $\varphi \in \mathscr{L}\text{-}\mathscr{P}$, then

$$L_1(x, \varphi^{(k)}(x)) \geq 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Pólya and Schur [1914] completely characterized classical multiplier sequences. Their seminal theorem maintains relevance in the setting of Legendre multiplier sequences, since every Legendre multiplier sequence must also be a classical multiplier sequence (see [Blakeman et al. 2012, Theorem 8] together with [Piotrowski 2007, Proposition 118]). We note that if $\{\gamma_k\}_{k=0}^{\infty}$ is a classical multiplier sequence, then one of

$$\{\gamma_k\}_{k=0}^{\infty}, \quad \{-\gamma_k\}_{k=0}^{\infty}, \quad \{(-1)^k \gamma_k\}_{k=0}^{\infty}, \quad \{(-1)^{k+1} \gamma_k\}_{k=0}^{\infty}$$

is a sequence of nonnegative terms [Pólya and Schur 1914, p. 90]. Since

$$\{-1\}_{k=0}^{\infty} \quad \text{and} \quad \{(-1)^k\}_{k=0}^{\infty}$$

are both classical multiplier sequences, it suffices to consider only sequences of nonnegative terms when characterizing classical multiplier sequences.

Theorem 5 [Pólya and Schur 1914]. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of nonnegative real numbers. The following are equivalent*:

(1) *$\{\gamma_k\}_{k=0}^{\infty}$ is a classical multiplier sequence.*
(2) *For each n, the polynomial $T[(1+x)^n] := \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$ is in $\mathscr{L}\text{-}\mathscr{P}^+$.*
(3) *$T[e^x] := \sum_{k=0}^{n} (\gamma_k / k!) x^k$ is in $\mathscr{L}\text{-}\mathscr{P}^+$.*

Similar to the classical setting, we may consider only sequences of nonnegative terms when investigating (linear and cubic) Legendre multiplier sequences, by virtue of $\{(-1)^k\}_{k=0}^{\infty}$ also being a Legendre multiplier sequence [Blakeman et al. 2012, Theorem 12].

We conclude this section by a theorem of Borcea and Brändén, which characterizes reality preserving linear operators $T : \mathbb{R}[x] \to \mathbb{R}[x]$ in terms of their symbol

$G_T(x, y)$. In order to be able to state their result (see Theorem 8), we need to make the following definitions.

Definition 6. The symbol of a linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is the formal power series defined by

$$G_T(x, y) := \sum_{n=0}^{\infty} \frac{(-1)^n T(x^n)}{n!} y^n.$$

Definition 7. A real polynomial $p \in \mathbb{R}[x, y]$ is called stable if $p(x, y) \neq 0$ whenever $\mathrm{Im}(x) > 0$ and $\mathrm{Im}(y) > 0$. The Laguerre–Pólya class of real entire functions in two variables, denoted by $\mathcal{L}\text{-}\mathcal{P}_2(\mathbb{R})$, is the set of real entire functions in two variables, which are locally uniform limits in $\mathbb{C}^2$ of real stable polynomials.

Theorem 8 [Borcea and Brändén 2009]. *A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ preserves the reality of zeros if and only if*

(1) *the rank of T is at most 2 and T is of the form $T(P) = \alpha(P)Q + \beta(P)R$, where $\alpha, \beta : \mathbb{R}[x] \to \mathbb{R}$ are linear functionals and $Q + iR$ is a stable polynomial, or*

(2) $G_T(x, y) \in \mathcal{L}\text{-}\mathcal{P}_2(\mathbb{R})$, *or*

(3) $G_T(-x, y) \in \mathcal{L}\text{-}\mathcal{P}_2(\mathbb{R})$.

In the remainder of this paper we follow the literature by using the notation $T = \{\gamma_k\}_{k=0}^{\infty}$ to indicate the dual interpretation of a sequence as a linear operator and vice versa.

3. Linear Legendre sequences

We now reprove the nonexistence of linear Legendre multiplier sequences (see [Blakeman et al. 2012, Proposition 2]). Although the result is known, our proof is novel, and has the promise of being suitable for use when investigating Q-multiplier sequences in larger generality. The following definition and three lemmas serve as setup for Theorem 13.

Definition 9. We define a generalized hypergeometric function by

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right] := 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{x^n}{n!}, \tag{3-1}$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ denotes the rising factorial.

The convergence properties of the series on the right hand side of Equation (3-1) are discussed in detail in [Rainville 1960, Chapter 5]. Here we mention that if $p = 3$ and $q = 2$, then the series is absolutely convergent on $|x| = 1$ if

$$\Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0.$$

Lemma 10. *For all $n \in \mathbb{N}^{\geq 1}$, the generalized hypergeometric function*

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2}, -n, \ \frac{1}{2}+n \\ \frac{1}{4}, \ \ \frac{3}{4} \end{matrix} ; -x\right] $$

converges at $x = -1$ and satisfies the equation

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2}, -n, \ \frac{1}{2}+n \\ \frac{1}{4}, \ \ \frac{3}{4} \end{matrix} ; 1\right] = 4n+1. $$

Proof. Convergence at $x = -1$ follows from the fact that

$$ \Re\left(\tfrac{1}{4}+\tfrac{3}{4}-\left(-\tfrac{1}{2}-n+\tfrac{1}{2}+n\right)\right) = 1 > 0, $$

together with the remark after Definition 9. The rest of the claim follows directly from an application of Theorem 30 in [Rainville 1960], which states that for nonnegative integers n, and a, b independent of n, we have

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}a-b, \ -n, \ a+n \\ 1+a-b, \ \frac{1}{2}a+\frac{1}{2} \end{matrix} ; 1\right] = \frac{(b)_n}{(1+a-b)_n}. $$

Setting $a = \frac{1}{2}$ and $b = \frac{5}{4}$ gives the required result. □

Lemma 11. *Let $n \in \mathbb{N}^{\geq 1}$, and define*

$$ \Psi_n(x) := \sum_{j=1}^{n} \binom{n}{j} \frac{(2j-2)!}{(j-1)!} \frac{\left(\frac{1}{2}+2j\right)_{n-j}}{\left(\frac{1}{2}\right)_n} x^j. $$

Then

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2}, -n, \ \frac{1}{2}+n \\ \frac{1}{4}, \ \ \frac{3}{4} \end{matrix} ; -x\right] = 1-2\Psi_n(x). $$

Proof. The following identities are readily verified for $0 \leq k \leq n$.

$$ (-1)^k \frac{(-n)_k}{k!} = \binom{n}{k}; \tag{3-2} $$

$$ \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k = \left(\frac{1}{2}\right)_{2k} 2^{-2k}; \tag{3-3} $$

$$ 2^k \left(-\frac{1}{2}\right)_k = -\frac{(2k-2)!}{2^{k-1}(k-1)!}; \tag{3-4} $$

$$ \frac{\left(\frac{1}{2}+n\right)_k}{\left(\frac{1}{2}\right)_{2k}} = \frac{\left(\frac{1}{2}+2k\right)_{n-k}}{\left(\frac{1}{2}\right)_n}. \tag{3-5} $$

With these in hand, we may now compute directly:

$$\begin{aligned}
{}_3F_2\left[\begin{matrix} -\frac{1}{2}, -n, \frac{1}{2}+n \\ \frac{1}{4}, \frac{3}{4} \end{matrix}; -x\right] &= \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_k (-n)_k \left(n+\frac{1}{2}\right)_k}{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k k!} (-x)^k \\
&= 1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{\left(n+\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_{2k} 2^{-2k}} (-x)^k \\
&= 1 - \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{\left(\frac{1}{2}+2k\right)_{n-k} 2^k (2k-2)!}{\left(\frac{1}{2}\right)_n 2^{k-1} (k-1)!} (-x)^k \\
&= 1 - 2\sum_{k=1}^{n} \binom{n}{k} \frac{\left(\frac{1}{2}+2k\right)_{n-k} (2k-2)!}{\left(\frac{1}{2}\right)_n (k-1)!} x^k \\
&= 1 - 2\Psi_n(x),
\end{aligned}$$

where the second equality uses equations (3-2) and (3-3), while the third equality employs equations (3-4) and (3-5). □

Lemma 12. *Let $C_n := \binom{2n}{n}/(n+1)$ denote the n-th Catalan number. For $n \in \mathbb{N}^{\geq 1}$ the following equality holds*:

$$-\frac{C_{n-1}}{3 \cdot 2^{2n-2}\left(\frac{5}{2}\right)_{2n-2}} = \frac{1}{2^{2n}\left(\frac{1}{2}\right)_{2n}} \left(2n \frac{(-1)^n \left(\frac{1}{2}\right)_n}{n!} + \sum_{j=1}^{n-1} \frac{C_{j-1}}{3 \cdot 2^{2j-2}\left(\frac{5}{2}\right)_{2j-2}} \frac{(-1)^{n-j}\left(\frac{1}{2}\right)_{n+j} 2^{2j}}{(n-j)!} \right).$$

Proof. Note that the statement of the lemma is equivalent to

$$0 = 2n \frac{(-1)^n \left(\frac{1}{2}\right)_n}{n!} + \sum_{j=1}^{n} \frac{C_{j-1} (-1)^{n-j} \left(\frac{1}{2}\right)_{n+j}}{\left(\frac{1}{2}\right)_{2j} (n-j)!} \quad \text{for all } n \in \mathbb{N}^{\geq 1}, \tag{3-6}$$

or

$$0 = 2n + \Psi_n(-1) \quad \text{for all } n \in \mathbb{N}^{\geq 1}, \tag{3-7}$$

where $\Psi_n(x)$ is as in Lemma 11. Combining the results of Lemmas 10 and 11 gives

$$1 - 2\Psi_n(-1) = 4n + 1 \quad \text{for all } n \in \mathbb{N}^{\geq 1},$$

or equivalently, $\Psi_n(-1) = -2n$ for $n \geq 1$. The proof is complete. □

We now prove the main theorem of the section.

Theorem 13. *Consider the operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ given by*

$$T[\mathfrak{Le}_k(x)] = (k+c)\mathfrak{Le}_k(x) \quad \textit{for } k = 0, 1, 2, 3, \dots \textit{ and } c \in \mathbb{R}.$$

If we write $T = \sum_{k=0}^{\infty} T_k(x) D^k$, *then*

$$T_k(0) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ c & \text{if } k = 0, \\ -\dfrac{C_{n-1}}{3 \cdot 2^{2n-2}\left(\frac{5}{2}\right)_{2n-2}} & \text{if } k = 2n \text{ for } n \geq 1, \end{cases} \tag{3-8}$$

where C_n *denotes the* n*-th Catalan number.*

Proof. The following facts about Legendre polynomials are known explicitly, or follow easily from basic properties (see [Rainville 1960, pp. 157–158]):

(i) $$\mathfrak{Le}_n(x) = \frac{2^n\left(\frac{1}{2}\right)_n x^n}{n!} + \pi_{n-2} \quad (n \geq 0),$$

where π_{n-2} is a polynomial of degree $n-2$ in x.

(ii) $$\mathfrak{Le}_{2n+1}(0) = 0, \text{ for } n \geq 0.$$

(iii) $$\mathfrak{Le}_{2n}(0) = \frac{(-1)^n\left(\frac{1}{2}\right)_n}{n!} \quad (n \geq 0).$$

(iv) For $0 \leq j \leq n$,

$$D^{2j}\,\mathfrak{Le}_{2n}(x)\big|_{x=0} = \frac{(-1)^{n-j}\left(\frac{1}{2}\right)_{n+j}2^{2j}}{(n-j)!},$$

while

$$D^{2j}\,\mathfrak{Le}_{2n+1}(x)\big|_{x=0} = 0 \quad \text{for all } j, n \geq 0,$$

because Legendre polynomials with odd index are odd.

Mutatis mutandis, the proof of Proposition 29 in [Piotrowski 2007] demonstrates that the coefficient polynomials $T_k(x)$ of the linear operator given in Theorem 13 can be computed recursively as

$$T_0(x) = T[1],$$
$$T_k(x) = \frac{1}{2^k\left(\frac{1}{2}\right)_k}\left(T[\mathfrak{Le}_k(x)] - \sum_{j=0}^{k-1} T_j(x) D^j[\mathfrak{Le}_k(x)]\right) \quad \text{for } k = 1, 2, 3, \ldots.$$

It is now easy to verify that $T_0(x) = c,\ T_1(x) = x$ and $T_2(x) = -\frac{1}{3}$, and the proposed values of $T_k(0)$ follow readily for $k = 0, 1, 2$. Proceeding by induction we assume that $T_j(0)$ is given by Equation (3-8) for $0 \leq j \leq k-1$ for some $k \geq 1$. If k is odd, the second part of fact (iv) above yields

$$\begin{aligned}
T_k(0) &= \frac{1}{2^k\left(\frac{1}{2}\right)_k}\left[(k+c)\mathfrak{Le}_k(0) - \sum_{j=0}^{k-1} T_j(x)D^j[\mathfrak{Le}_k(x)]\Big|_{x=0}\right] \\
&= \frac{1}{2^k\left(\frac{1}{2}\right)_k}\left[-\sum_{j=0}^{(k-1)/2} T_{2j}(x)D^{2j}[\mathfrak{Le}_k(x)]\Big|_{x=0}\right] \\
&= 0.
\end{aligned}$$

On the other hand, if k is even, writing $k = 2n$ and using the first part of fact (iv) gives

$$\begin{aligned}
T_k(0) &= \frac{1}{2^{2n}\left(\frac{1}{2}\right)_{2n}}\left[(2n+c)\frac{(-1)^n\left(\frac{1}{2}\right)_n}{n!} - \sum_{j=0}^{k-1} T_j(x)D^j[\mathfrak{Le}_k(x)]\Big|_{x=0}\right] \\
&= \frac{1}{2^{2n}\left(\frac{1}{2}\right)_{2n}}\left[2n\frac{(-1)^n\left(\frac{1}{2}\right)_n}{n!} - \sum_{j=1}^{k-1} T_j(x)D^j[\mathfrak{Le}_k(x)]\Big|_{x=0}\right] \\
&= \frac{1}{2^{2n}\left(\frac{1}{2}\right)_{2n}}\left[2n\frac{(-1)^n\left(\frac{1}{2}\right)_n}{n!} - \sum_{j=1}^{(k-2)/2} T_{2j}(x)D^{2j}[\mathfrak{Le}_k(x)]\Big|_{x=0}\right] \\
&= \frac{1}{2^{2n}\left(\frac{1}{2}\right)_{2n}}\left[2n\frac{(-1)^n\left(\frac{1}{2}\right)_n}{n!} + \sum_{j=1}^{n-1} \frac{C_{j-1}}{3\cdot 2^{2j-2}\left(\frac{5}{2}\right)_{2j-2}}\,\frac{(-1)^{n-j}\left(\frac{1}{2}\right)_{n+j}2^{2j}}{(n-j)!}\right] \\
&= -\frac{C_{n-1}}{3\cdot 2^{2n-2}\left(\frac{5}{2}\right)_{2n-2}},
\end{aligned}$$

where the last equality is the result of Lemma 12. □

Let T be the operator corresponding to the Legendre sequence $\{k+c\}_{k=0}^{\infty}$. Recall that the symbol of T is given by

$$G_T(-x, y) = \sum_{k=0}^{\infty} \frac{(-1)^k T[x^k]y^k}{k!},$$

and that T is reality preserving (that is, $\{k+c\}_{k=0}^{\infty}$ is a Legendre multiplier sequence) if and only if either $G_T(-x, y)$ or $G_T(x, y)$ belongs to $\mathcal{L}$-$\mathcal{P}_2(\mathbb{R})$, since the sequence under consideration is nontrivial. Following [Brändén and Ottergren 2014], we expand $G_T(-x, y)$ and $G_T(x, y)$ as a series in powers of x. By Theorem 13 the constant term in both of these expansions is

$$f(y) := c - \sum_{k=1}^{\infty} \frac{C_{k-1}y^{2k}}{3\cdot 2^{2k-2}\left(\frac{5}{2}\right)_{2k-2}}.$$

Thus $f(y) \in \mathscr{L}\text{-}\mathscr{P}$ if either $G_T(-x, y)$ or $G_T(x, y)$ were in $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$, since we obtain $f(y)$ from either $G_T(-x, y)$ or $G_T(x, y)$ by applying the nonnegative multiplier sequence $1, 0, 0, 0, \dots$ acting on x, which preserves the class $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ (see [Borcea and Brändén 2010; Brändén 2014]). We shall now demonstrate that $f(y)$ is an entire function which does not belong to the Laguerre–Pólya class, and hence $\{k+c\}_{k=0}^{\infty}$ is not a Legendre multiplier sequence for any $c \in \mathbb{R}$.

Proposition 14. *Let $c \in \mathbb{R}$. Then*

$$f(y) = c - \sum_{k=1}^{\infty} \frac{C_{k-1} y^{2k}}{3 \cdot 2^{2k-2} \left(\frac{5}{2}\right)_{2k-2}}$$

is an entire function which does not belong to $\mathscr{L}\text{-}\mathscr{P}$.

Proof. Consider the change of variables $x = y^2$ and the function

$$\tilde{f}(x) = c - \frac{4}{3} \sum_{k=1}^{\infty} \frac{C_{k-1} x^k}{2^{2k} \left(\frac{5}{2}\right)_{2k-2}} = c - \frac{4}{3} \sum_{k=1}^{\infty} a_k x^k.$$

Since

$$(\star) \qquad \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2(2k-1)}{k+1} \frac{1}{(5+2(2k-2))(5+2(2k-1))} = 0,$$

$\tilde{f}(x)$ is entire. The existence of the limit in $(\star)$ implies that $\lim_{k\to\infty} \sqrt[k]{a_k} = 0$ as well, and hence $f(y)$ is also entire.

It remains to show that $f(y) \notin \mathscr{L}\text{-}\mathscr{P}$. To this end, we first demonstrate that $\tilde{f}(x) \notin \mathscr{L}\text{-}\mathscr{P}$. Writing $d_k = k!\, a_k$ we can express $\tilde{f}(x)$ as

$$\tilde{f}(x) = c - \frac{4}{3} \sum_{k=1}^{\infty} \frac{d_k}{k!} x^k.$$

By Theorem 4 and the comments thereafter, if $\tilde{f}(x)$ were to belong to $\mathscr{L}\text{-}\mathscr{P}$, we would have $L_1(x, \tilde{f}^{(k)}) \geq 0$ for all $k = 0, 1, 2, \dots$ and $x \in \mathbb{R}$. In particular, $L_1(0, \tilde{f}') = \frac{16}{9}(d_2^2 - d_3 d_1) \geq 0$ would hold. A quick calculation reveals that

$$d_2^2 - d_3 d_1 = -\tfrac{1}{80850} < 0,$$

establishing that $\tilde{f}(x) \notin \mathscr{L}\text{-}\mathscr{P}$. Suppose now that $f(y) \in \mathscr{L}\text{-}\mathscr{P}$. By virtue of being an even function, $f(y)$ has the factorization

$$f(y) = c e^{-ay^2} \prod_{k=1}^{\omega} \left(1 - \frac{y^2}{x_k^2}\right),$$

where $a \geq 0$, $x_k \in \mathbb{R} \setminus \{0\}$, $0 \leq \omega \leq \infty$, and $\sum 1/x_k^2 < \infty$. Replacing y^2 by x would yield $\tilde{f}(x) \in \mathcal{L}\text{-}\mathcal{P}$, a contradiction. We conclude that $f(y) \notin \mathcal{L}\text{-}\mathcal{P}$, and our proof is complete. ☐

4. Cubic Legendre multiplier sequences

In this section we establish the nonexistence of cubic Legendre multiplier sequences. Without loss of generality we may consider sequences interpolated by monic polynomials. Since every such cubic polynomial can be written as $(k^2+\alpha k+\beta)(k+c)$ for some real triple (α, β, c), one may wish to proceed based on whether or not the quadratic factor in the product is itself a Legendre multiplier sequence. It turns out that such case analysis is more than one needs: we can handle all cubic sequences at once. We begin with two preparatory results.

Lemma 15. *Suppose $T = \{k^3+ak^2+bk+c\}_{k=0}^{\infty}$ is a sequence of nonnegative terms. If T is a classical multiplier sequence, then $a \geq -3$, $a+b \geq -1$ and $c \geq 0$.*

Proof. By Theorem 5, T is a classical multiplier sequence if and only if $T[e^x] \in \mathcal{L}\text{-}\mathcal{P}^+$. We have

$$T[e^x] = \sum_{k=0}^{\infty}(k^3+ak^2+bk+c)\frac{x^k}{k!}$$
$$= e^x\big(x^3+(a+3)x^2+(a+b+1)x+c\big).$$

Thus the coefficients of the polynomial

$$x^3+(a+3)x^2+(a+b+1)x+c$$

must all be nonnegative. The claim follows. ☐

Lemma 16 [Levin 1956, Lemma 3, p. 337]. *If all zeros of the real polynomial*

$$h(x) = c_0 + c_1 x + \cdots + c_n x^n \quad (c_n \neq 0)$$

are real, $c_0 \neq 0$ and $c_p = 0$ for some $0 < p < n$, then $c_{p-1}c_{p+1} < 0$.

We are now ready to state and prove the main theorem of the section.

Theorem 17. *The sequence $\{k^3+ak^2+bk+c\}_{k=0}^{\infty}$ is not a Legendre multiplier sequence for any real triple (a, b, c).*

Proof. Denote by $T_{a,b,c}$ the operator associated to the Legendre sequence

$$\{k^3+ak^2+bk+c\}_{k=0}^{\infty}.$$

By Lemma 15, in order for $\{k^3+ak^2+bk+c\}_{k=0}^{\infty}$ to be a classical multiplier sequence we must have $a \geq -3$, $a+b \geq -1$ and $c \geq 0$. Consider now the action of $T_{\alpha,\beta,c}$ on the two polynomials

$$\begin{aligned}p_1(x) &= x^5\mathfrak{Le}_3(x)\\ &= \tfrac{64}{1287}\mathfrak{Le}_8(x)+\tfrac{152}{693}\mathfrak{Le}_6(x)+\tfrac{372}{1001}\mathfrak{Le}_4(x)+\tfrac{205}{693}\mathfrak{Le}_2(x)+\tfrac{4}{63},\\ p_2(x) &= x^5\mathfrak{Le}_5(x)\\ &= \tfrac{2016}{46189}\mathfrak{Le}_{10}(x)+\tfrac{4816}{24453}\mathfrak{Le}_8(x)+\tfrac{4078}{11781}\mathfrak{Le}_6(x)\\ &\quad+\tfrac{291}{1001}\mathfrak{Le}_4(x)+\tfrac{1000}{9009}\mathfrak{Le}_2(x)+\tfrac{8}{693}.\end{aligned}$$

Computing $18018T_{a,b,c}[p_1(x)]=\sum_{k=0}^{4}q_{2k}(a,b,c)x^{2k}$, we find that

$$\begin{aligned}q_0(a,b,c) &= 16(-121+46a-46b),\\ q_4(a,b,c) &= 630(15724+1226a+61b),\end{aligned}$$

with the restrictions on a, b and c implying directly that $q_4(a,b,c)>0$ for all real triples (a,b,c) under consideration. If $q_2(a,b,c)=0$, then reversing coefficients, and taking four derivatives of $T_{a,b,c}[p_1(x)]$ (both of which operations preserve the reality of zeros) results in a polynomial with nonreal zeros. If $q_2(a,b,c)\neq 0$, then in light of Lemma 16, a necessary condition for $T_{a,b,c}[p_1(x)]$ to have only real zeros is that

$$(\dagger)\qquad q_0(a,b,c)=16(-121+46a-46b)\geq 0.$$

We now turn our attention to $T_{a,b,c}[p_2(x)]$. If we write $23279256T_{a,b,c}[p_2(x)]=\sum_{k=0}^{5}w_{2k}(a,b,c)x^{2k}$, then

$$\begin{aligned}w_0(a,b,c) &= 16(-641+806a-806b),\\ w_4(a,b,c) &= -630(38840980+2015774a+62731b),\end{aligned}$$

with Lemma 15 implying that $w_4(a,b,c)<0$ for all admissible triples (a,b,c). Considerations identical to those above imply that either $T_{a,b,c}[p_2(x)]$ has nonreal zeros, or the inequality

$$(\ddagger)\qquad w_0(a,b,c)=16(-641+806a-806b)\leq 0$$

must hold. Combining inequalities $(\dagger)$ and $(\ddagger)$ we obtain

$$-\tfrac{121}{46}+a\geq b\geq -\tfrac{641}{806}+a,$$

a clear impossibility. We conclude that $T_{a,b,c}$ cannot simultaneously preserve the reality of the zeros of $x^5\mathfrak{Le}_3(x)$ and $x^5\mathfrak{Le}_5(x)$. Whence $\{k^3+ak^2+bk+c\}_{k=0}^{\infty}$ is not a Legendre multiplier sequence for any real triple (a,b,c). □

Remark 18. Theorem 17 yields yet another proof of the nonexistence of linear Legendre multiplier sequences by the following considerations. If T_1, T_2 are Legendre multiplier sequences, then so is T_1T_2. Since $\{k^2+k+\beta\}$ is a Legendre multiplier sequence whenever $\beta \in [0, 1]$, the existence of linear Legendre multiplier sequences would immediately imply the existence of cubic Legendre multiplier sequences, contradicting Theorem 17.

5. Open problems

The following is a list of open problems motivated by the preceding results. These questions are not only related to the classification of Legendre multiplier sequences but also to some general properties of reality preserving linear operators $T = \sum_{k=0}^{\infty} T_k(x) D^k$ on $\mathbb{R}[x]$, properties which are captured in the coefficient polynomials $T_k(x)$.

5.1. ***Higher order Legendre sequences.*** The characterization of polynomials with degree four or higher which interpolate Legendre multiplier sequences remains open. Using computational techniques as in Section 4 quickly turns intractable with the increasing number of parameters. In addition, one has to judiciously select "test polynomials" in order for this method to succeed succinctly. The polynomials

$$p(n, k) = x^k \mathfrak{Le}_n(x)$$

mimic properties of the test polynomials $(1+x)^n$ for classical multiplier sequences in that they have zeros of high multiplicity away from the zeros of the basis polynomials. As such, we were able to use just a couple test polynomials to demonstrate the nonexistence of cubic Legendre multiplier sequences. On the downside, the degrees of these polynomials are high and we believe that the degrees of the test polynomials would have to increase if one would want to eliminate sequences interpolated by higher order polynomials.

5.2. ***Monotone operators.*** We call an operator $T = \sum_{k=0}^{\infty} T_k(x) D^k$ monotone if $\deg T_k(x) \geq \deg T_{k-1}(x)$ for all $k = 1, 2, \ldots$. The operator corresponding to the linear Legendre sequence $\{k+c\}_{k=0}^{\infty}$ is given by

$$T = c + xD - \frac{1}{3}D^2 + \frac{2}{15}xD^3 + \sum_{k=4}^{\infty} T_k(x) D^k,$$

whereas the operator corresponding to the Legendre sequence $\{k^2+\alpha k+\beta\}_{k=0}^{\infty}$, $\alpha \neq 1$, is given by

$$\begin{aligned} T = \beta + (1+\alpha)xD - \frac{2+\alpha-3x^2}{3}D^2 + \frac{2}{15}(\alpha-1)xD^3 \\ - \frac{(\alpha-1)(1+4x^2)}{105}D^4 + (\alpha-1)\sum_{k=5}^{\infty} T_k(x) D^k. \end{aligned}$$

Neither sequence is a Legendre multiplier sequence, and neither operator is monotone. We believe these facts to be related, and give the following

Conjecture 19. *Suppose $T = \sum_{k=0}^{\infty} T_k(x) D^k$ is an infinite order differential operator. If T is not monotone, then T is not reality preserving.*

Should this conjecture be true, one could then try to prove that if

$$\{\gamma_k\}_{k=0}^{\infty} = \{p(k)\}_{k=0}^{\infty},$$

where deg p is odd and $\{\gamma_k\}_{k=0}^{\infty}$ is a Legendre sequence, the operator corresponding to the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is an infinite order differential operator which is not monotone.

5.3. ***Using the symbol of the operator.*** Our approach used in Section 3 could be extended to treat sequences interpolated by higher order polynomials. Piotrowski [2007] and Forgács and Piotrowski [2013a] give explicit representations of the coefficient polynomials $T_k(x)$ of classical, and Hermite diagonal operators respectively. In both cases the $T_k(x)$s are given in terms of the reverses of the Jensen polynomials associated to the sequence $\{\gamma_k\}_{k=0}^{\infty}$. If a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is interpolated by a polynomial, then only finitely many of these reverse Jensen polynomials are nonzero. This means that an analog of Theorem 13 would need the identification of only finitely many sequences, one for each reverse Jensen polynomial involved in the $T_k(x)$s, in order to explicitly determine the sequence $\{T_k(0)\}_{k=0}^{\infty}$. With this sequence in hand, one could carry out steps analogous to those in Section 3 to establish the nonexistence of Legendre multiplier sequences interpolated by polynomials of degree greater than three.

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