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A not-so-simple Lie bracket expansion

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Lie algebras and quantum groups are not usually studied by an undergraduate. However, in the study of these structures, there are interesting questions that are easily accessible to an upper-level undergraduate. Here we look at the expansion of a nested set of brackets that appears in relations presented in a paper of Lum on toroidal algebras. We illuminate certain terms that must be in the expansion, providing a partial answer for the closed form.

1. Introduction

Lie algebras and quantum groups are not topics that you are apt to hear undergraduates math majors discussing in their spare time. However, there are a surprising number of nontrivial questions in this area that are undergraduate appropriate. In this paper, we will give a brief overview of the broad mathematical setting, and then discuss an accessible problem that involves expanding a nested set of brackets.

Lie algebras, their universal enveloping algebras and quantum groups are a fundamental part of representation theory that have many applications within mathematics and mathematical physics. Lie algebras and Lie groups were originally discovered by Sophus Lie in the late nineteenth century [Borel 2001]. Given a Lie algebra, we associate a unique associative algebra called the universal enveloping algebra. In 1985, Jimbo and Drinfeld discovered q-analogues of these universal enveloping algebras called "quantum groups", which have been a recent area of study (see [Lusztig 1993]).

In order to find the quantum analogue of a Lie algebra it is often desirable to understand the defining relationships of the Lie algebra inside of its universal enveloping algebra. The motivation for this project came from a paper by Lum in which he gives a nice presentation of a toroidal Lie algebra that could be useful in understanding this Lie algebra's quantum group [Lum 1998]. All of these relations utilize a nested set of brackets called $t(k)$. For simplicity, we have modified $t(k)$ by a scalar. In this paper we seek to understand the expansion of this object.

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2. The Lie bracket $t(k)$

Recall that a *Lie algebra* is defined as a vector space L over a field F that is equipped with a bilinear map $L \times L \to L$, known as a *Lie bracket*, satisfying certain conditions. The Lie bracket $(x, y) \to [x, y]$ for all $x, y \in L$ must satisfy the *alternating property*, namely

$$[x, x] = 0$$

and the *Jacobi identity*, an analog of associativity:

$$\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0,$$

for all $x, y, z \in L$.

Example 2.1. The set $gl(n, \mathbb{R})$ of n-by-n matrices with real entries, together with the operation defined by defined by $[A, B] := AB - BA$, is a Lie algebra. To see this, consider matrices $A, B, C \in gl(n, \mathbb{R})$. It is easy to show that the bracket is bilinear. Since $[A, A] = AA - AA = 0$ alternation is satisfied. To verify that the Jacobi identity holds, note that

$$\begin{aligned}
\big[A, [B, C]\big] + \big[B, [C, A]\big] + \big[C, [A, B]\big]& \\
&= [A, BC - CB] + [B, CA - AC] + [C, AB - BA] \\
&= A(BC - CB) - (BC - CB)A + B(CA - AC) \\
&\qquad - (CA - AC)B + C(AB - BA) - (AB - BA)C \\
&= ABC - ACB - BCA + CBA + BCA - BAC \\
&\qquad - CAB + ACB + CAB - CBA - ABC + BAC \\
&= 0.
\end{aligned}$$

The nested set of brackets which we seek to understand is denoted $t(k)$. They are defined recursively as

$$t(1) := [x, y] = xy - yx, \quad t(k) := \underbrace{\big[\ldots[[[x, y], x], y]\ldots\big]}_{k\ xy\text{-pairs}}$$

Example 2.2. The case of $k = 3$ is given as follows:

$$\begin{aligned}
t(3) &= \big[[[[[x, y], x], y], x], y\big] \\
&= \big[[[[xy - yx, x\big], y], x], y\big] \\
&= \big[[2xyxy - 2yxyx + y^2x^2 - x^2y^2, x], y\big] \\
&= 4xyxyxy - 4yxyxyx + 2y^2xyx^2 - 2x^2yxy^2 + 2yx^2yxy - 2yxyx^2y \\
&\qquad + y^2x^3y - yx^3y^2 + yx^2y^2x - xy^2x^2y + yxy^2x^2 - x^2y^2xy + x^3y^3 - y^3x^3.
\end{aligned}$$

We choose to view the output of $t(k)$ as *words*. Unlike the combinatorial definition of words, we include the coefficient. For example, $t(1)$ consists of two words, namely xy and $-yx$. We know some additional properties of words because of how the bracket functions. No word can begin and end with an x nor can a word begin and end with y^2.

We define the *antiword* to be the associated word with reverse ordering of x's and y's, opposite sign, and same coefficient. In the example of $t(1)$, the antiword of xy is $-yx$. Similarly, $-yx$ has antiword xy. For a more interesting example, consider Example 2.2. Notice that each word is written next to its antiword. We have $4xyxyxy$ followed by $-4yxyxyx$ which is a word-antiword pair, $2y^2xyx^2$ followed by $-2x^2yxy^2$, and so on. This observation works in general and cuts the problem in half.

Theorem 2.3. *Every word in $t(k)$ has an antiword in $t(k)$.*

Proof. For the base case $k = 1$, we have $t(1) := [x, y] = xy - yx$, so the statement is clearly valid. Now assume for some integer $k \geq 1$ every word appears in $t(k)$ together with its antiword. We want to show that each word in the $t(k+1)$ has an antiword in $t(k+1)$. So consider an arbitrary word ω and its antiword $\overline{\omega}$ in the k-th iteration. By bracket expansion, we have

$$\begin{aligned}\big[[\omega+\overline{\omega}, x], y\big] &= [\omega x + \overline{\omega}x - x\omega - x\overline{\omega}, y] \\ &= \omega xy + \overline{\omega}xy - x\omega y - x\overline{\omega}y - (y\omega x + y\overline{\omega}x - yx\omega - yx\overline{\omega}) \\ &= \omega xy + \overline{\omega}xy - x\omega y - x\overline{\omega}y - y\omega x - y\overline{\omega}x + yx\omega + yx\overline{\omega}.\end{aligned}$$

Since ω and $\overline{\omega}$ are a word-antiword pair, the following are word-antiword pairs: ωxy and $yx\overline{\omega}$, $-x\omega y$ and $-y\overline{\omega}x$, $-y\omega x$ and $-x\overline{\omega}y$, and $yx\omega$ and $\overline{\omega}xy$. Each of these words will have the same coefficient as ω and $\overline{\omega}$. If two of these words in $t(k+1)$ are the same, the words in $t(k)$ that generated them have corresponding antiwords in $t(k)$. Bracketing these will necessarily give the same antiword in $t(k+1)$ causing coefficients to be preserved. Thus, while a whole pair may cancel, no word can independently disappear. □

From this result, we know that it is not possible to have any symmetric words in the output of an arbitrary $t(k)$.

3. Word patterns

Given our goal to determine the content of $t(k)$ in the universal enveloping algebra, we first look to locate patterns universal to all $t(k)$. Here we prove the existence of several such patterns of words. First we consider two fundamental lemmas.

Lemma 3.1. *If the word $x^k y^k$ exists in $t(k)$, it must be generated by the word $x^{k-1}y^{k-1}$ in $t(k-1)$. Similarly, if the word $y^k x^k$ exists in $t(k)$, it must be generated by the word $y^{k-1}x^{k-1}$ in $t(k-1)$.*

Proof. Assume the word $x^k y^k$ exists in $t(k)$. By the definition of the bracket, in order to arrive at this word, we must multiply some word in $t(k-1)$ by both an x and a y. Working backwards, we remove a y and an x in all possible ways to obtain possible root words for $x^k y^k$. Our only option is to remove a y from the end and an x from the beginning. Therefore our only root word is $x^{k-1}y^{k-1}$. Showing that $y^k x^k$ is only generated by the root word $y^{k-1}x^{k-1}$ is analogous. □

The lemma below follows in an identical fashion.

Lemma 3.2. *If the word $(xy)^k$ exists in $t(k)$, it must be generated by $(xy)^{k-1}$ or $(yx)^{k-1}$ in $t(k-1)$. Similarly, if the word $(yx)^k$ exists in $t(k)$ it must be generated by $(xy)^{k-1}$ or $(yx)^{k-1}$ in $t(k-1)$.*

The proceeding propositions use these lemmas to show some universal patterns appearing in $t(k)$ for all k.

Proposition 3.3. *The words $(-1)^{k+1}x^k y^k$ and $(-1)^k y^k x^k$ appear in $t(k)$.*

Proof. These words appear in the case of $k = 1$ since

$$[x, y] = xy - yx = (-1)^2 xy + (-1)yx.$$

Assume that for some integer $k \geq 1$, we have the words $(-1)^{k+1}x^k y^k + (-1)^k x^k y^k$. We now show that the words $(-1)^{k+2}x^{k+1}y^{k+1}$ and $(-1)^{k+1}x^{k+1}y^{k+1}$ appear in $t(k+1)$. By the definition of the bracket, we have

$$\begin{aligned}
&\big[[(-1)^{k+1}x^k y^k, x], y\big] \\
&\quad = \big[(-1)^{k+1}x^{k+1}y^k - (-1)^{k+1}x^k y^k x, y\big] \\
&\quad = \big[(-1)^{k+1}x^{k+1}y^k + (-1)^{k+2}x^k y^k x, y\big] \\
&\quad = (-1)^{k+1}yx^{k+1}y^k + (-1)^{k+2}yx^k y^k x - \big((-1)^{k+1}x^{k+1}y^{k+1} + (-1)^{k+2}x^k y^k xy\big) \\
&\quad = (-1)^{k+1}yx^{k+1}y^k + (-1)^{k+2}yx^k y^k x + (-1)^{k+2}x^{k+1}y^{k+1} + (-1)^{k+3}x^k y^k xy.
\end{aligned}$$

The word $(-1)^{k+2}x^{k+1}y^{k+1}$ appears as desired. It is an identical process to prove the existence of $(-1)^{k+1}y^{k+1}x^{k+1}$. Furthermore, we know from Lemma 3.1 that $x^{k+1}y^{k+1}$ and $y^{k+1}x^{k+1}$ cannot be generated by any other root words. Therefore the coefficient is as given. □

Using this same technique we find two more words that appear in $t(k)$.

Proposition 3.4. *The words $2^{k-1}(xy)^k$ and $-2^{k-1}(yx)^k$ appear in $t(k)$.*

4. More general recurring words

We now look to find broader patterns of words which necessarily appear in $t(k)$. Similar to before, we need a foundational lemma.

Lemma 4.1. *For $k \geq 1$ and $2 \leq j \leq k$, the word $x^j(yx)^{k-j}y^j$, if it exists in $t(k)$, can only be generated by the word $x^{j-1}(yx)^{((k-1)-(j-1))}y^{j-1}$ in $t(k-1)$ and the word $y^j(xy)^{k-j}x^j$, if it exists in $t(k)$, can only be generated by the word $y^{j-1}(xy)^{((k-1)-(j-1))}x^{j-1}$ in $t(k-1)$.*

In order to prove this lemma, we use similar techniques to that of the previous lemmas. We begin by assuming that the words appear in the k-th iteration of the bracket and we work backwards to determine possible root words. This relatively simple procedure is all that is needed to show that the lemma holds. Using this lemma, we now expand the notions of Proposition 3.3 and Proposition 3.4.

Theorem 4.2. *For $k \geq 1$ and $1 \leq j \leq k$, the word $-\tau_k^j x^j(yx)^{k-j}y^j + \tau_k^j y^j(xy)^{k-j}x^j$ appears in $t(k)$, where we have set*

$$\tau_k^j := (-1)^j 2^{k-j}.$$

To prove this more encompassing theorem, we use double induction. We know that this theorem holds for the base case $k = 1$ and $j = 1$

$$[x, y] = xy - yx = (-1)^2 2^0 x(yx)^0 y + (-1)^1 2^0 y(xy)^0 x$$

and by Proposition 3.4, we know the statement holds for arbitrary k and $j = 1$. Subsequently, we use this as a starting point for the second induction. Simply use a bracket argument similar to the one in Proposition 3.3. This argument yields all of the desired words except in the case of $j = k$. However, Proposition 3.3 already accounts for this case. Therefore, the statement is satisfied.

Returning to our running example of $k = 3$, notice that Theorem 4.2 asserts the existence of the following words: $4xyxyxy$, $-4yxyxyx$, $2y^2xyx^2$, $-2x^2yxy^2$, x^3y^3, and $-y^3x^3$. In Example 2.2, we see that all of these do indeed appear in $t(3)$.

This collection of words accounts for a share of the words in $t(k)$. Unfortunately, it does not even account for all of the words in the case of $k = 3$. However, repeated bracketing of words in Theorem 4.2 will result in more words that are always present. We leave showing the following corollary by bracket as an exercise.

Corollary 4.3. *For $k \geq 1$ and $1 \leq j \leq k$, the following sum appears in $t(k+1)$:*

$$\begin{aligned}
&-\tau_k^j x^j(yx)^{k-j}y^j xy + \tau_k^j y^j(xy)^{k-j}x^{j+1}y + \tau_k^j x^{j+1}(yx)^{k-j}y^{j+1} \\
&-\tau_k^j xy^j(xy)^{k-j}x^j y + \tau_k^j yx^j(yx)^{k-j}y^j x - \tau_k^j y^{j+1}(xy)^{k-j}x^{j+1} \\
&-\tau_k^j yx^{j+1}(yx)^{k-j}y^j + \tau_k^j yxy^j(xy)^{k-j}x^j.
\end{aligned}$$

Indeed, the words from Corollary 4.3 actually include all of the words in Theorem 4.2 as shown below.

Proposition 4.4. *All words in $t(k+1)$ of the form $-\tau_k^{j+1} x^{j+1}(yx)^{k-(j+1)} y^{j+1}$ and $\tau_k^{j+1} y^{j+1}(xy)^{k-(j+1)} x^{j+1}$ can be expressed by a form given in Corollary 4.3.*

Proof. Consider the word in $t(k+1)$ generated by Theorem 4.2 given by

$$(-1)^{(j+1)+1} 2^{(k+1)-(j+1)} x^{j+1}(yx)^{(k+1)-(j+1)} y^{j+1} \;=\; \tau_k^j x^{j+1}(yx)^{k-j} y^{j+1}.$$

This word is also a word of the form given in Corollary 4.3. Furthermore, consider the other word in $t(k+1)$ generated by Theorem 4.2:

$$(-1)^{j+1} 2^{(k+1)-(j+1)} y^{j+1}(xy)^{(k+1)-(j+1)} x^{j+1} \;=\; -\tau_k^j y^{j+1}(xy)^{k-j} x^{j+1}$$

which is indeed a word of the desired form. These two general words account for all words of the form given by Theorem 4.2 in $t(k+1)$ except for the case of $j=1$.

First consider $(-1)^{j+1} 2^{(k+1)-j} x^j (yx)^{(k+1)-j} y^j$ generated by Theorem 4.2 evaluated at $j=1$. This yields

$$(-1)^2 2^k x(yx)^k y = 2^k (xy)^{k+1} = 2^{k-1} x(yx)^{k-1} yxy + 2^{k-1} xy(xy)^{k-1} xy$$

which are two words in Corollary 4.3 evaluated at $j=1$, namely $-\tau_k^j x^j (yx)^{k-j} y^j xy$ and $-\tau_k^j xy^j (xy)^{k-j} x^j y$. The proof that $(-1)^j 2^{(k+1)-j} y^j (xy)^{(k+1)-j} x^j$ can be expressed in a desired form when $j=1$ is identical. □

Using Proposition 4.4, we account for all of the words in $t(1)$, $t(2)$, and $t(3)$. We leave showing that Corollary 4.3 produces all of $t(3)$ as an exercise. Moreover, we believe that we can identify an even larger pattern of words.

As seen in previous cases, we first identify how the particular words can be generated.

Proposition 4.5. *If the words in the left column of the table below exist in $t(k)$, they must be generated by the corresponding root word listed on the right.*

Generated word in $t(k)$	*Root word in $t(k-1)$*
$y^m x^j (yx)^{k-(j+m)} y^j x^m$	$y^{m-1} x^j (yx)^{(k-1)-(j+(m-1))} y^j x^{m-1}$
$x^m y^j (xy)^{k-(j+m)} x^j y^m$	$x^{m-1} y^j (xy)^{k-(j+(m-1))} x^j y^{m-1}$
$x^j (yx)^{k-(j+m)} y^j (xy)^m$	$x^j (yx)^{k-(j+(m-1))} y^j (xy)^{m-1}$
$(yx)^m y^j (yx)^{k-(j+m)} x^j$	$(yx)^{m-1} y^j (yx)^{k-(j+(m-1))} x^j$
$(yx)^m x^j (yx)^{k-(j+m)} y^j$	$(yx)^{m-1} x^j (yx)^{k-(j+(m-1))} y^j$
$y^j (xy)^{k-(j+m)} x^j (xy)^m$	$y^j (xy)^{k-(j+(m-1))} x^j (xy)^{m-1}$

Despite the larger number of words in question, the proof of each follows in the same manner as our previous proofs for necessary root words, except in the more complicated case of $m = 1$. In this instance, there are more ways to remove one x and y. However, it can be shown that some of these violate the properties of words and thus do not exist in $t(k+1)$.

Building from all of our previous work we present our largest list of necessary words.

Theorem 4.6. *Let $k \geq 1$.*

- *If $k \geq 2$, $j = 1$, and $m = 1$, then*
$$\tau_k^{j+m}(yx)^m x^j (yx)^{k-(j+m)} y^j - \tau_k^{j+m} y^j (xy)^{k-(j+m)} x^j (xy)^m$$
appears in $t(k)$.
- *If $1 \leq j \leq k$ and $m = 0$, then*
$$-\tau_k^{j+m} y^m x^j (yx)^{k-(j+m)} y^j x^m + \tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m$$
appears in $t(k)$.
- *If $m \geq 1$, $j \geq 2$, and $j + m \leq k$, then*
$$\begin{aligned}&-\tau_k^{j+m} y^m x^j (yx)^{k-(j+m)} y^j x^m + \tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m \\ &\quad + (-1)^m\big(-\tau_k^{j+m} x^j (yx)^{k-(j+m)} y^j (xy)^m + \tau_k^{j+m}(yx)^m y^j (yx)^{k-(j+m)} x^j \\ &\qquad - \tau_k^{j+m}(yx)^m x^j (yx)^{k-(j+m)} y^j + \tau_k^{j+m} y^j (xy)^{k-(j+m)} x^j (xy)^m\big)\end{aligned}$$
appears in $t(k)$.

Proof. Proof of the $j = 1$, $m = 1$ case follows directly from Corollary 4.3 by evaluating $-\tau_k^j yx^{j+1}(yx)^{k-j} y^j$ at $j = 1$.

Now consider the case of $m = 0$. We have
$$(-1)^{j+0+1} 2^{k-(j+0)} y^0 x^j (yx)^{k-(j+0)} y^j x^0 = -\tau_k^j x^j (yx)^{k-j} y^j.$$
This word was shown to exist in $t(k)$ by Theorem 4.2. For the same reason, we know that $\tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m$ exists in $t(k)$ when $m = 0$.

Now we show $-\tau_k^{j+m} y^m x^j (yx)^{k-(j+m)} y^j x^m$ and $\tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m$ appear in $t(k)$ if $j \geq 2$ and $3 \leq m + j \leq k$. We just argued the case of $m = 0$ for arbitrary $1 \leq j \leq k$ for all $t(k)$. So, we perform induction on m. In Corollary 4.3, we bracket $-\tau_k^j x^j (yx)^{k-j} y^j + (-1)^j 2^{k-j} y^j (xy)^{k-j} x^j$ in $t(k)$ with $k \geq 2$ to generate the term $(-1)^{j+2} 2^{k-j} yx^j (yx)^{k-j} y^j x - \tau_k^j xy^j (xy)^{k-j} x^j y$ which is the desired term for $m = 1$ in $t(k+1)$.

Now, assume that $k \geq 3$ and that for some $m \geq 1$ with $m + j \leq k$, the words $-\tau_k^{j+m} y^m x^j (yx)^{k-(j+m)} y^j x^m + \tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m$ appear in $t(k)$. We want to show that
$$(-1)^{j+(m+1)+1} 2^{k-(j+(m+1))} y^{m+1} x^j (yx)^{k-(j+(m+1))} y^j x^{m+1}$$

$$+(-1)^{j+(m+1)}2^{(k+1)-(j+(m+1))}x^{m+1}y^j(xy)^{(k+1)-(j+(m+1))}x^jy^{m+1}$$

appears in $t(k+1)$. Using our bracket, we have

$$\begin{aligned}
&\big[[-\tau_k^{j+m}y^mx^j(yx)^{k-(j+m)}y^jx^m+\tau_k^{j+m}x^my^j(xy)^{k-(j+m)}x^jy^m,x],y\big]\\
&\quad=\big[-\tau_k^{j+m}y^mx^j(yx)^{k-(j+m)}y^jx^{m+1}+\tau_k^{j+m}x^my^j(xy)^{k-(j+m)}x^jy^mx\\
&\qquad\quad+\tau_k^{j+m}xy^mx^j(yx)^{k-(j+m)}y^jx^m-\tau_k^{j+m}x^{m+1}y^j(xy)^{k-(j+m)}x^jy^m,y\big]\\
&\quad=-\tau_k^{j+m}y^mx^j(yx)^{k-(j+m)}y^jx^{m+1}y+\tau_k^{j+m}x^my^j(xy)^{k-(j+m)}x^jy^mxy\\
&\qquad\quad+\tau_k^{j+m}xy^mx^j(yx)^{k-(j+m)}y^jx^my-\tau_k^{j+m}x^{m+1}y^j(xy)^{k-(j+m)}x^jy^{m+1}\\
&\qquad\quad+\tau_k^{j+m}y^{m+1}x^j(yx)^{k-(j+m)}y^jx^{m+1}+\tau_k^{j+m}yx^my^j(xy)^{k-(j+m)}x^jy^mx\\
&\qquad\quad-\tau_k^{j+m}yxy^mx^j(yx)^{k-(j+m)}y^jx^m+\tau_k^{j+m}yx^{m+1}y^j(xy)^{k-(j+m)}x^jy^m.
\end{aligned}$$

We have the resulting words (line 3 word 1 and line 2 word 2)

$$\begin{aligned}
&\tau_k^{j+m}y^{m+1}x^j(yx)^{k-(j+m)}y^jx^{m+1}\\
&\qquad=(-1)^{j+(m+1)+1}2^{(k+1)-(j+(m+1))}y^{m+1}x^j(yx)^{(k+1)-(j+(m+1))}y^jx^{m+1}
\end{aligned}$$

and

$$\begin{aligned}
&-\tau_k^{j+m}x^{m+1}y^j(xy)^{k-(j+m)}x^jy^{m+1}\\
&\qquad=(-1)^{j+(m+1)}2^{(k+1)-(j+(m+1))}x^{m+1}y^j(xy)^{(k+1)-(j+(m+1))}x^jy^{m+1}.
\end{aligned}$$

By Proposition 4.5, we know that these words cannot be generated by any other root word. The other four remaining desired words can be shown through an analogous process. □

5. Moving forward

We could consider continuing our current course of action by looking for new patterns beginning in the $k = 4, 5, 6$ cases to try to detect another significant margin of words. One difficulty with this avenue is that an entirely new class of words appears every few cases. A second difficulty is that these become time consuming for the computer to compute. Maple 15 was unable to compute these brackets at $t(8)$ after a full day of computation for $t(7)$. It appears that every time this version of Maple encounters a noncommuting term like xyx it computes $x * y * x$. However, Sage (sagemath.org) treats xyx as an element and can compute the values much faster. Despite this, at $t(11)$ it starts to take minutes for the computation to occur, and it is expected that even using SAGE the computational time would be too high before reaching $t(20)$.

The reader may be wondering why we have taken this particular approach to the problem. The answer is quite simple. We have been unable to find a nice

k	word count	k	word count	k	word count
4	46	7	1648	10	61512
5	152	8	5506	11	206028
6	500	9	18380	12	691126

Table 1. Number of words in $t(k)$.

combinatorial method to simplify the problem. The number of terms in $t(k)$ grows rapidly; see Table 1. Our initial use of dominoes, strips, and tableaux illustrated interesting connections but did not yield useful results. Then we used the Online Encyclopedia of Integer Sequences (oeis.org) to try and find connections to other less obvious options. However, despite searching a number of related sequences, we were unable to locate any connections. It would be ideal if one could find such a connection in order to continue this problem.

This problem is thus still open, as is the question of expanding the full relations given in Lum's paper. We encourage readers to improve on our method and find connections to solve these problems. After this is done, it will be possible to give a nice presentation of the toroidal quantum group.

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