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a journal of mathematics

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2013 vol. 6, no. 4

INVOLVE 6:4 (2013)
dx.doi.org/10.2140/involve.2013.6.383

Embeddedness for singly periodic Scherk surfaces with higher dihedral symmetry

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(Communicated by Frank Morgan)

The singly periodic Scherk surfaces with higher dihedral symmetry have $2n$-ends that come together based upon the value of φ. These surfaces are embedded provided that $\frac{\pi}{2} - \frac{\pi}{n} < \frac{n-1}{n}\varphi < \frac{\pi}{2}$. Previously, this inequality has been proved by turning the problem into a Plateau problem and solving, and by using the Jenkins–Serrin solution and Krust's theorem. In this paper we provide a proof of the embeddedness of these surfaces by using some results about univalent planar harmonic mappings from geometric function theory. This approach is more direct and explicit, and it may provide an alternate way to prove embeddedness for some complicated minimal surfaces.

1. Introduction

A minimal surface in $\mathbb{R}^3$ is a surface whose mean curvature vanishes at each point on the surface. One area of minimal surface theory that has seen a lot of interest and results recently is the study of complete embedded minimal surfaces. Minimal surfaces can be parametrized by the classical Weierstrass representation. However, these surfaces are not guaranteed to be complete and embedded. In this paper we will consider the family of singly periodic Scherk surfaces with higher dihedral symmetry that were first described in the seminal paper [Karcher 1988]. They belong to the larger class of embedded singly periodic minimal surfaces with Scherk ends and genus 0 in the quotient that have been completely classified in [Pérez and Traizet 2007]. The singly periodic Scherk surfaces with higher dihedral symmetry have $2n$-ends that come together based upon the value of φ. In particular, it was shown in [Weber 2005] that these surfaces are embedded provided that

$$\frac{\pi}{2} - \frac{\pi}{n} < \frac{n-1}{n}\varphi < \frac{\pi}{2}. \tag{1}$$

MSC2010: 30C45, 49Q05, 53A10.
Keywords: minimal surfaces, harmonic mappings, Scherk, univalence.
Part of this research was done during the 2010 BYU REU supported by NSF grant DMS-0755422.

Previously, this inequality has been established by turning the problem into a Plateau problem and solving, and by using the Jenkins–Serrin solution and Krust's theorem. In this paper, we will provide a proof of the embeddedness of these surfaces by using some results about univalent planar harmonic mappings from geometric function theory. This approach is more direct and explicit, and it may provide an alternate way to prove embeddedness for some complicated minimal surfaces. In the interesting paper [McDougall and Schaubroeck 2008], the authors discuss similar harmonic mappings and the corresponding minimal surfaces. They also work to prove an inequality similar to (1). While their approach is sound, there are unfortunately several small mistakes and errors, and the inequality they give is incorrect and different from the result in [Weber 2005]. In our paper, we start with planar harmonic mappings but then approach the proof of the inequality in a different way and derive the correct inequality given by (1).

This approach involves the following steps. First, we will construct a φ-variable family of planar harmonic functions that map the unit disk univalently onto a $2n$-gon region. Next, we will compute the value of φ for which these functions are convex. Then, we will use a simple convolution theorem to construct a "conjugate" family of planar harmonic functions that are also univalent. Finally, using a Weierstrass representation we will lift this last family to minimal graphs that turn out to be the singly periodic Scherk surfaces with higher dihedral symmetry. Because of the harmonic functions are univalent, the embeddedness of the Scherk surfaces is guaranteed.

2. A family of univalent planar harmonic mappings

Definition 2.1. A continuous function $f(x, y) = u(x, y) + i v(x, y)$ defined in a domain $G \subset \mathbb{C}$ is a *complex-valued harmonic function* in G if u and v are real harmonic functions in G.

Complex-valued harmonic functions defined on $\mathbb{D}$, the unit disk, are related to analytic functions, as the following theorem shows.

Theorem 2.2 [Clunie and Sheil-Small 1984]. *If $f = u + i v$ is harmonic in a simply connected domain G, then f can be written as $f = h + \bar{g}$, where h and g are analytic.*

We are interested in univalent (one-to-one) harmonic mappings. While it is often difficult to establish the univalency of a planar harmonic function, we do have the following nice result about local univalency.

Lemma 2.3 [Lewy 1936]. *The harmonic function $f = h + \bar{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|g'(z)/h'(z)| < 1$ for all $z \in \mathbb{D}$.*

The function $\omega(z) = g'(z)/h'(z)$ is known as the dilatation and plays an important role in the theory of univalent harmonic mappings.

We will now consider a specific family of planar harmonic mappings that are related to Scherk surfaces. Let $f_n(z) = h_n(z) + \overline{g_n(z)}$ be the family of planar harmonic mappings from $\mathbb{D}$ into $\mathbb{C}$, where

$$h_n'(z) = \frac{1}{(z^n - e^{i\varphi})(z^n - e^{-i\varphi})}, \quad g_n'(z) = \frac{z^{2n-2}}{(z^n - e^{i\varphi})(z^n - e^{-i\varphi})},$$

$n \geq 2$ and $\varphi \in [0, \frac{\pi}{2}]$. Thus,

$$f_n(z) = \int_0^z \frac{d\zeta}{(\zeta^n - e^{i\varphi})(\zeta^n - e^{-i\varphi})} + \overline{\int_0^z \frac{\zeta^{2n-2}\, d\zeta}{(\zeta^n - e^{i\varphi})(\zeta^n - e^{-i\varphi})}}.$$

Note that $g_n'(z)/h_n'(z) = z^{2n-2}$. Letting ξ be the primitive n-th root of unity and using the residue theorem, we can compute that

$$\begin{aligned} h_n(z) =& \frac{1}{2n \sin\varphi} \int_0^z \left(\sum_{j=1}^n \frac{-i e^{-i(\frac{n-1}{n})\varphi} \xi^j}{\zeta - e^{i\frac{\varphi}{n}} \xi^j} + \sum_{j=1}^n \frac{i e^{i(\frac{n-1}{n})\varphi} \xi^j}{\zeta - e^{-i\frac{\varphi}{n}} \xi^j} \right) d\zeta \\ =& \frac{1}{2n \sin\varphi} \sum_{k=1}^n \Big(-i e^{-i(\frac{n-1}{n}\varphi + \frac{2k\pi}{n})} \log\big(z - e^{i(\frac{\varphi}{n} - \frac{2k\pi}{n})}\big) \\ & \qquad + i e^{i(\frac{n-1}{n}\varphi + \frac{2k\pi}{n})} \log\big(z - e^{-i(\frac{\varphi}{n} - \frac{2k\pi}{n})}\big) \Big). \end{aligned}$$

Similarly,

$$\begin{aligned} g_n(z) =& \frac{1}{2n \sin\varphi} \sum_{k=1}^n \Big(-i e^{i(\frac{n-1}{n}\varphi + \frac{2k\pi}{n})} \log\big(z - e^{i(\frac{\varphi}{n} - \frac{2k\pi}{n})}\big) \\ & \qquad + i e^{-i(\frac{n-1}{n}\varphi + \frac{2k\pi}{n})} \log\big(z - e^{-i(\frac{\varphi}{n} - \frac{2k\pi}{n})}\big) \Big). \end{aligned}$$

Since $f_n(z) = \operatorname{Re}(h_n(z) + g_n(z)) + i \operatorname{Im}(h_n(z) - g_n(z))$, after normalizing so that $f_n(0) = 0$, we get

$$\begin{aligned} f_n(z) = \frac{1}{n \sin\varphi} \sum_{k=1}^n \Big\{ & \cos\left(\frac{n-1}{n}\varphi + \frac{2k\pi}{n}\right)\left(\beta_1 - \beta_2 + \frac{4k\pi}{n}\right) \\ & - i \sin\left(\frac{n-1}{n}\varphi + \frac{2k\pi}{n}\right)(\beta_1 + \beta_2) \Big\}, \end{aligned} \tag{2}$$

where

$$\beta_1 = \arg\big(z + e^{i(\frac{\varphi}{n} - \frac{2k\pi}{n})}\big),$$
$$\beta_2 = \arg\big(z + e^{-i(\frac{\varphi}{n} - \frac{2k\pi}{n})}\big).$$

Theorem 2.4. *The harmonic function f_n maps $\mathbb{D}$ onto a $2n$-gon.*

Because the dilatation $\omega_n(z)$ equals $g_n'(z)/h_n'(z) = z^{2n-2}$, we know that f_n maps arcs of $\partial\mathbb{D}$ to either concave arcs or to stationary points [Bshouty and Hengartner 1997; Bshouty et al. 2008]. Letting $z = e^{i\theta} \in \partial\mathbb{D}$, we see that the latter situation occurs. In particular, f_n maps $\partial\mathbb{D}$ to vertices, v_m ($m = 1, \dots, 2n$), of a $2n$-gon such that

$$\arg v_m = e^{\frac{i(j-1)\pi}{n}} \quad \text{and} \quad |v_m| = \begin{cases} |v_1| & \text{if } v_m \text{ is odd,} \\ |v_2| & \text{if } v_m \text{ is even,} \end{cases}$$

where it can be computed that

$$v_1 = \frac{\pi}{n \sin\varphi}\left(\cos\frac{(n-1)\varphi}{n} + \cot\frac{\pi}{n}\sin\frac{(n-1)\varphi}{n}\right) + 0i, \tag{3}$$

$$v_2 = \frac{\pi}{n \sin\varphi}\sin\frac{(n-1)\varphi}{n}\left(\cot\frac{\pi}{n} + i\right). \tag{4}$$

Example 2.5. For $n = 4$, we have

$$f_4(z) = \operatorname{Re}(h_4(z) + g_4(z)) + i\,\operatorname{Im}(h_4(z) - g_4(z)),$$

where

$$\begin{aligned}\operatorname{Re}(h_4(z)+g_4(z)) = \frac{1}{4\sin\varphi}\Big(\cos\frac{3\varphi}{4}\big[&\arg\big(z-e^{i\frac{\varphi}{4}}\big) - \arg\big(z+e^{i\frac{\varphi}{4}}\big) \\ &- \arg\big(z-e^{-i\frac{\varphi}{4}}\big) + \arg\big(z+e^{-i\frac{\varphi}{4}}\big)\big] \\ + \sin\frac{3\varphi}{4}\big[&\arg\big(z-e^{i(\frac{\varphi}{4}+\frac{\pi}{2})}\big) - \arg\big(z+e^{i(\frac{\varphi}{4}+\frac{\pi}{2})}\big) \\ &+ \arg\big(z-e^{-i(\frac{\varphi}{4}-\frac{\pi}{2})}\big) - \arg\big(z+e^{-i(\frac{\varphi}{4}-\frac{\pi}{2})}\big)\big]\Big) \\ + \frac{2\pi}{4\sin\varphi}\Big(\cos\frac{3\varphi}{4} + \sin\frac{3\varphi}{4}\Big)&\end{aligned}$$

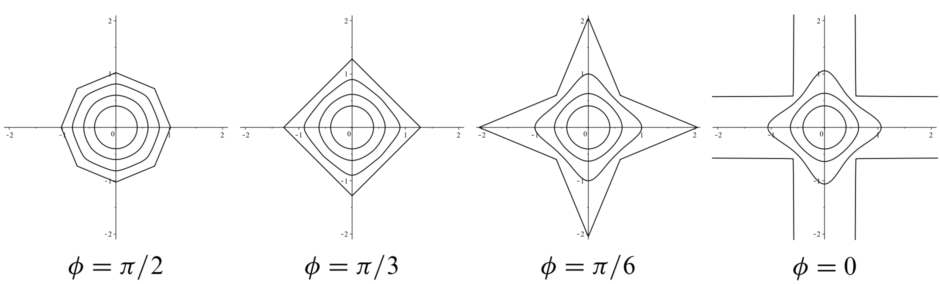


Figure 1. Images under f_4 of concentric circles in $\mathbb{D}$ for various values of φ.

and

$$\operatorname{Im}(h_4(z)-g_4(z)) = \frac{1}{4\sin\varphi}\Bigg(\sin\frac{3\varphi}{4}\big[-\arg\big(z-e^{i\frac{\varphi}{4}}\big)+\arg\big(z+e^{i\frac{\varphi}{4}}\big) - \arg\big(z-e^{-i\frac{\varphi}{4}}\big)+\arg\big(z+e^{-i\frac{\varphi}{4}}\big)\big] + \cos\frac{3\varphi}{4}\big[\arg\big(z-e^{i(\frac{\varphi}{4}+\frac{\pi}{2})}\big)+\arg\big(z-e^{i(\frac{\varphi}{3}+\frac{2\pi}{3})}\big) - \arg\big(z-e^{-i(\frac{\varphi}{4}-\frac{\pi}{2})}\big)+\arg\big(z+e^{-i(\frac{\varphi}{4}-\frac{\pi}{2})}\big)\big]\Bigg).$$

Letting

$$M = \frac{\pi}{4\sin\varphi}\cos\frac{3\varphi}{4} \quad \text{and} \quad N = \frac{\pi}{4\sin\varphi}\sin\frac{3\varphi}{4},$$

we see that f_4 maps $\partial\mathbb{D}$ to the vertices of an octagon as follows (see Figure 1):

$$f_4(e^{i\theta}) = \begin{cases} v_1 = (M+N) & \text{if } -\frac{\varphi}{4} < \theta < \frac{\varphi}{4}, \\ v_2 = N + iN & \text{if } \frac{\varphi}{4} < \theta < \frac{\pi}{2} - \frac{\varphi}{4}, \\ v_3 = i(M+N) & \text{if } \frac{\pi}{2} - \frac{\varphi}{4} < \theta < \frac{\pi}{2} + \frac{\varphi}{4}, \\ v_4 = -N + iN & \text{if } \frac{\pi}{2} + \frac{\varphi}{4} < \theta < \pi - \frac{\varphi}{4}, \\ v_5 = -(M+N) & \text{if } \pi - \frac{\varphi}{4} < \theta < \pi + \frac{\varphi}{4}, \\ v_6 = -N - iN & \text{if } \pi + \frac{\varphi}{4} < \theta < \frac{3\pi}{2} - \frac{\varphi}{4}, \\ v_7 = -i(M+N) & \text{if } \frac{3\pi}{2} - \frac{\varphi}{4} < \theta < \frac{3\pi}{2} + \frac{\varphi}{4}, \\ v_8 = N - iN & \text{if } \frac{3\pi}{2} + \frac{\varphi}{4} < \theta < -\frac{\varphi}{4}. \end{cases}$$

Theorem 2.6. *For $n \geq 2$, f_n is univalent for all $z \in \mathbb{D}$ and $\varphi \in (0, \frac{\pi}{2}]$.*

Proof. This follows from a result by Duren, McDougall and Schaubroeck [Duren et al. 2005] that states if a harmonic function f is of the form (2) constructed with a piecewise constant boundary function and with values on the m vertices of a polygonal region Ω and with $\omega = g'(z)/h'(z)$ being a Blaschke product with at most $m-2$ factors, then

$$f(z) \text{ is univalent in } \mathbb{D} \iff \text{all the zeros of } \omega \text{ lie in } \mathbb{D}. \qquad \square$$

Remark 2.7. For $n = 3, 4$, one can simply employ the shearing technique of Clunie and Sheil-Small [1984] to prove univalency with even less background. However, for $n \geq 5$ the shearing technique cannot be applied to f_n.

Theorem 2.8. *The image $f_n(\mathbb{D})$ is convex for every $\varphi \in \left(\frac{n}{n-1}\left(\frac{\pi}{2} - \frac{\pi}{n}\right), \frac{\pi}{2}\right]$.*

Proof. Note that f_n will be convex for every φ if

$$\operatorname{Re} v_2 > \tfrac{1}{2}\operatorname{Re}(v_1 + v_3) \quad \text{and} \quad \operatorname{Im} v_2 > \tfrac{1}{2}\operatorname{Im}(v_1 + v_3).$$

From (3), it is clear that

$$\begin{aligned}
\operatorname{Re} v_1 &= v_1, & \operatorname{Im} v_1 &= 0,\\
\operatorname{Re} v_2 &= v_1 - \frac{\pi \cos \frac{(n-1)\varphi}{n}}{n \sin \varphi}, & \operatorname{Im} v_2 &= \frac{\pi \sin \frac{(n-1)\varphi}{n}}{n \sin \varphi},\\
\operatorname{Re} v_3 &= \operatorname{Re}(e^{i\frac{2\pi}{n}} v_1) = \cos \frac{2\pi}{n}\, v_1, & \operatorname{Im} v_3 &= \operatorname{Im}(e^{i\frac{2\pi}{n}} v_1) = \sin \frac{2\pi}{n}\, v_1.
\end{aligned}$$

Setting $\operatorname{Re} v_2 = \frac{1}{2}\operatorname{Re}(v_1 + v_3)$ and solving for v_1 yields

$$v_1 = \frac{2\pi}{n} \cdot \frac{\cos \frac{(n-1)\varphi}{n}}{\sin \varphi \left(1 - \cos \frac{2\pi}{n}\right)}. \tag{5}$$

Likewise, setting $\operatorname{Im}(v_2) = \frac{1}{2}\operatorname{Im}(v_1 + v_3)$ and again solving for v_1 yields

$$v_1 = \frac{2\pi}{n} \cdot \frac{\sin \frac{(n-1)\varphi}{n}}{\sin \varphi \sin \frac{2\pi}{n}}. \tag{6}$$

Equating (5) and (6) and solving for φ we obtain

$$\varphi = \frac{n}{n-1} \arctan \frac{\sin \frac{2\pi}{n}}{1 - \cos \frac{2\pi}{n}} = \frac{n}{n-1}\left(\frac{\pi}{2} - \frac{\pi}{n}\right). \qquad \square$$

There is a convolution theorem for planar harmonic mappings that takes univalent convex maps and transforms them into new harmonic maps while preserving univalency. We will apply this convolution theorem to those functions f_n that map $\mathbb{D}$ onto a convex domain. But first, we need some background. For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad F(z) = \sum_{n=0}^{\infty} A_n z^n,$$

their convolution is defined as

$$f(z) * F(z) = \sum_{n=0}^{\infty} a_n A_n z^n.$$

Note that the right half-plane mapping, $f(z) = z/(1-z)$, acts as the convolution identity; that is, if F is an analytic function, then

$$\frac{z}{1-z} * F(z) = F(z).$$

Now let's consider the case of harmonic convolutions.

Definition 2.9. Given harmonic univalent functions

$$f(z) = h(z) + \bar{g}(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n,$$

$$F(z) = H(z) + \overline{G}(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \bar{z}^n,$$

define the *harmonic convolution* as

$$f(z) * F(z) = h(z) * H(z) + \overline{g(z) * G(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \bar{z}^n.$$

Lemma 2.10 [Clunie and Sheil-Small 1984]. *Let $f = h + \bar{g}$ be a harmonic univalent mapping from $\mathbb{D}$ onto a convex domain and normalized so that $f(0) = 0$ and $f_z(0) = 1$. Also, let ϕ be a normalized univalent analytic function from $\mathbb{D}$ onto a convex domain. Then for* $(|\alpha| \le 1)$,

$$f \, * \, (\alpha\bar{\phi} + \phi) = h * \phi + \alpha\overline{g * \phi}$$

is a univalent harmonic map $\mathbb{D}$ onto a close-to-convex domain.

Theorem 2.11. *The function F_n is univalent on $\mathbb{D}$ for $\varphi \in \left(\frac{n}{n-1}\left(\frac{\pi}{2} - \frac{\pi}{n}\right), \frac{\pi}{2}\right]$.*

Proof. From Theorem 2.8, we know the f_n are convex maps for $\frac{n}{n-1}\left(\frac{\pi}{2} - \frac{\pi}{n}\right) < \varphi \le \frac{\pi}{2}$. Hence for these values of φ we can apply Lemma 2.10 with $\phi = z/(1-z)$ and $\alpha = -1$ to create the planar harmonic maps

$$F_n(z) = \mathrm{Re}\,(h_n(z) - g_n(z)) + i\,\mathrm{Im}\,(h_n(z) + g_n(z))$$

which are univalent in $\mathbb{D}$. □

Example 2.12. From Theorem 2.11, we conclude that the harmonic maps $F_4(z)$ are univalent in $\mathbb{D}$ (see Figure 2).

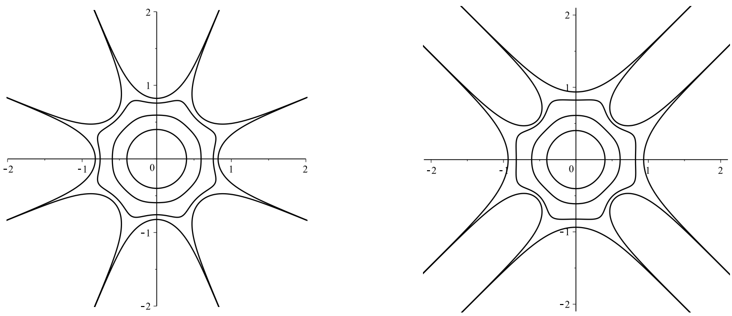
3. Singly periodic Scherk surfaces with higher dihedral symmetry

The connection between planar harmonic mappings and minimal surfaces can be seen in the following *Weierstrass representation* (see [Duren 2004], for example):

Theorem 3.1. *Let $f = h + \bar{g}$ be an orientation-preserving harmonic univalent mapping of $\mathbb{D}$ onto some domain Ω with dilatation $\omega = q^2$, where q is an analytic function in $\mathbb{D}$. Then*

$$X(z) = \left(\mathrm{Re}(h(z) + g(z)),\ \mathrm{Im}(h(z) - g(z)),\ 2\,\mathrm{Im} \int_0^z \sqrt{g'(\zeta)h'(\zeta)}\, d\zeta\right)$$

gives an isothermal parametrization of a minimal graph whose projection in the xy-plane is f.



$\phi = \pi/2$ $\phi = \pi/3$

Figure 2. Images under F_4 of concentric circles in $\mathbb{D}$ for various values of φ.

Thus, univalent planar harmonic mappings with a dilatation that is the square of an analytic function lift to minimal graphs in $\mathbb{R}^3$. We have shown that both families f_n and F_n of harmonic mappings satisfy the hypotheses of Theorem 3.1 for a given range of φ values and will thus lift to embedded minimal graphs. To identify these surfaces, we use the following standard Weierstrass representation.

Theorem 3.2 (Weierstrass representation (G, dh) [Weber 2005]). *Every regular minimal surface has a local isothermal parametric representation of the form*

$$X(z) = \operatorname{Re} \int_a^z \left(\frac{1}{2}\left(\frac{1}{G} - G\right), \frac{i}{2}\left(\frac{1}{G} + G\right), 1 \right) dh,$$

where G is the Gauss map, dh is the height differential, and $a \in \mathbb{D}$ is a constant.

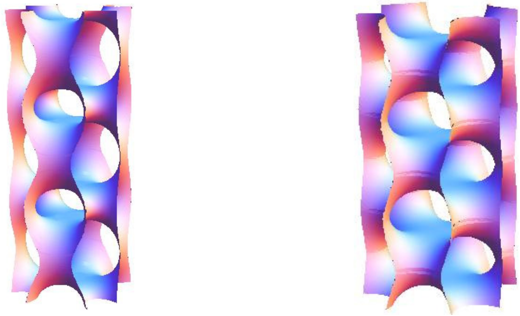
Proving the embeddedness of singly periodic Scherk surfaces with higher dihedral symmetry is not easy. However, with the material we have developed it follows naturally.

Theorem 3.3. *F_n lifts to a family of embedded singly periodic Scherk surfaces with higher dihedral symmetry for φ satisfying (1).*

Proof. Scalings and reflections across planes containing two axes do not alter the geometry of minimal surfaces. So we can use the coordinate functions from the two Weierstrass representations to get

$$h = \int_0^z \frac{1}{G}\, dh, \quad g = \int_0^z G\, dh. \tag{7}$$

In [Weber 2005] the Gauss map and height differential for a family of minimal surfaces ranging from Scherk's singly periodic surface with $2n$ ends when $\varphi = \frac{\pi}{2}$



$\phi = \pi/2$ $\phi = \pi/3$

Figure 3. Singly periodic Scherk surfaces.

to the n-noid when $\varphi = 0$ is given by

$$G = z^{n-1}, \quad dh = \frac{z^{n-1}}{(z^n - e^{i\varphi})(z^n - e^{-i\varphi})}.$$

Using the formulas in (7) we see

$$h^* = \int_0^z \frac{d\zeta}{(\zeta^n - e^{i\varphi})(\zeta^n - e^{-i\varphi})}, \quad g^* = -\int_0^z \frac{\zeta^{2n-2}\,d\zeta}{(\zeta^n - e^{i\varphi})(\zeta^n - e^{-i\varphi})}.$$

It is clear that $F_n = h^* + \overline{g^*}$. Hence, we see that F_n lifts to this family of singly periodic Scherk's surfaces for all $\varphi \in \left(\frac{n}{n-1}\left(\frac{\pi}{2} - \frac{\pi}{n}\right), \frac{\pi}{2}\right]$. □

Remark 3.4. We could have used Krust's theorem [Dierkes et al. 1992] instead of Lemma 2.10. But this convolution theorem is not well known and is a generalization of Krust's Theorem applied to planar harmonic mappings.

Remark 3.5. The harmonic maps, f_n, lift to a family of minimal surfaces that continuously transform from Scherk's first surface with $2n$-ends to a minimal surface with n-helicoidal ends. Because the harmonic maps are univalent, the resulting minimal surfaces are graphs. However, they are graphs only over the domain $\mathbb{D}$. This does not contradict the fact that the minimal surface with n helicoidal ends is not embedded since the surface is defined on a domain larger than $\mathbb{D}$.

Area for further investigation. Apply the approach used in this paper to prove the embeddedness for less symmetric Scherk-like surfaces and for the twist deformation of Scherk's singly periodic surfaces (see [Weber 2005, pp. 39–40]).

Acknowledgements

The authors would like to thank Casey Douglas for his comments and suggestions.

References

[Bshouty and Hengartner 1997] D. Bshouty and W. Hengartner, "Boundary values versus dilatations of harmonic mappings", *J. Anal. Math.* **72** (1997), 141–164. MR 99c:30061 Zbl 0908.30017

[Bshouty et al. 2008] D. Bshouty, A. Lyzzaik, and A. Weitsman, "On the boundary behaviour of univalent harmonic mappings onto convex domains", *Comput. Methods Funct. Theory* **8**:1-2 (2008), 261–275. MR 2010b:30069 Zbl 1160.30002

[Clunie and Sheil-Small 1984] J. Clunie and T. Sheil-Small, "Harmonic univalent functions", *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), 3–25. MR 85i:30014 Zbl 0506.30007

[Dierkes et al. 1992] U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab, *Minimal surfaces, I: Boundary value problems*, Grundl. Math. Wiss. **295**, Springer, Berlin, 1992. MR 94c:49001a Zbl 0777.53012

[Duren 2004] P. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics **156**, Cambridge University Press, 2004. MR 2005d:31001 Zbl 1055.31001

[Duren et al. 2005] P. Duren, J. McDougall, and L. Schaubroeck, "Harmonic mappings onto stars", *J. Math. Anal. Appl.* **307**:1 (2005), 312–320. MR 2006c:31002 Zbl 1112.31001

[Karcher 1988] H. Karcher, "Embedded minimal surfaces derived from Scherk's examples", *Manuscripta Math.* **62**:1 (1988), 83–114. MR 89i:53009 Zbl 0658.53006

[Lewy 1936] H. Lewy, "On the non-vanishing of the Jacobian in certain one-to-one mappings", *Bull. Amer. Math. Soc.* **42**:10 (1936), 689–692. MR 1563404 Zbl 0015.15903

[McDougall and Schaubroeck 2008] J. McDougall and L. Schaubroeck, "Minimal surfaces over stars", *J. Math. Anal. Appl.* **340**:1 (2008), 721–738. MR 2009d:31004 Zbl 1169.53007

[Pérez and Traizet 2007] J. Pérez and M. Traizet, "The classification of singly periodic minimal surfaces with genus zero and Scherk-type ends", *Trans. Amer. Math. Soc.* **359**:3 (2007), 965–990. MR 2007m:53010 Zbl 1110.53008

[Weber 2005] M. Weber, "Classical minimal surfaces in Euclidean space by examples: geometric and computational aspects of the Weierstrass representation", pp. 19–63 in *Global theory of minimal surfaces*, edited by D. Hoffman, Clay Math. Proc. **2**, Amer. Math. Soc., Providence, RI, 2005. MR 2006e:53025 Zbl 1100.53015

Received: 2012-05-23 Revised: 2012-07-24 Accepted: 2012-07-25

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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2013 vol. 6 no. 4