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The concept of frames in a Banach space has been introduced by Gröchenig and developed by several authors. The main feature of a frame is to present every element of the underlying Banach space as a norm-convergent series. In this decomposition, the dual frame plays an essential role. The existence of a dual p -frame is not guaranteed in general. Some characterizations of duals of p -frames are given in this paper.

1. Introduction and preliminaries

A sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space \mathcal{H} is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2 \quad (f \in \mathcal{H}). \quad (1-1)$$

The numbers A and B are called *frame bounds*. A frame is called *tight* if $A = B$. In frame theory, the operator $T : l^2 \rightarrow \mathcal{H}$ given by $T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i$ is useful in analyzing various properties of frames. It is called the *synthesis* or *preframe operator*. Its adjoint $T^* : \mathcal{H} \rightarrow l^2; f \mapsto \{\langle f, f_i \rangle\}_{i=1}^{\infty}$ is called the *analysis operator*. By composing T and T^* , we obtain the frame operator

$$S : \mathcal{H} \rightarrow \mathcal{H}, \quad Sf = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i \quad (f \in \mathcal{H}).$$

The frame operator S is invertible and the reconstruction formula

$$f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle f_i \quad (f \in \mathcal{H}) \quad (1-2)$$

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holds. The sequence $\{S^{-1}f_i\}_{i=1}^{\infty}$ which plays the same role as the dual in the theory of bases is also a frame. It is called the *canonical dual* of $\{f_i\}_{i=1}^{\infty}$. In general, the Bessel sequence $\{g_i\}_{i=1}^{\infty}$ is called a *dual* of $\{f_i\}_{i=1}^{\infty}$ if

$$f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i \quad (f \in \mathcal{H}). \quad (1-3)$$

For general references on this theory, we refer the reader to [Christensen 2008, Section 5.1]. Recently, various generalizations of frames have been proposed: continuous frames [Ali et al. 1993; Askari-Hemmat et al. 2001; Gabardo and Han 2003], g -frames [Sun 2006], fusion frames [Casazza et al. 2008], von Neumann–Schatten frames [Sadeghi and Arefijamaal 2012], and so on. Frames for Banach spaces were first introduced in [Gröchenig 1991] and were developed in [Aldroubi et al. 2001; Cazassa and Christensen 1997; Casazza et al. 1999; 2005]. In particular, Christensen and Stoeva [2003] studied p -frames in Banach spaces and obtained a lot of interesting and important results.

In applications of frame theory the goal is to recognize the finer properties of functions by means of the magnitudes of the frame coefficients [Benedetto et al. 2006; Bolcskei et al. 1998; Candès and Donoho 2004; Heath and Paulraj 2002]. These properties, typically smoothness and decay properties or phase-space localization of functions, are measured by the Banach space norm. Dual frames have a key role in the decomposition of elements in the underlying space. Casazza et al. [2005] present some equivalent conditions for the existence of reconstruction formulas in Banach spaces. Moreover, sufficient conditions for the existence of dual frames are studied in [Aldroubi et al. 2001]. In this article, at the first, we review the definition and basic properties of p -frames, and then express some characterizations of duals of p -frames. The analogous results concerning frames in Hilbert spaces may be found in [Li 1995]. Finally, we discuss a stability theorem for duals of p -frames.

2. Elementary properties of p -frames

Throughout this paper, X is a separable Banach space with dual X^* , $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A sequence $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is called a p -frame for X if there exist constants $A, B > 0$ such that

$$A\|x\| \leq \left(\sum_{i=1}^{\infty} |g_i(x)|^p \right)^{\frac{1}{p}} \leq B\|x\| \quad (x \in X). \quad (2-1)$$

The sequence $\{g_i\}_{i=1}^{\infty}$ is a p -Bessel sequence if at least the upper p -frame condition is satisfied. Analogous to frame theory in Hilbert spaces, one can define

the synthesis operator as

$$T : l^q \rightarrow X^*, \quad T\{d_i\} := \sum_{i=1}^{\infty} d_i g_i.$$

A straightforward calculation shows that $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p -Bessel sequence with bound B if and only if T is well-defined and $\|T\| \leq B$; see [Christensen and Stoeva 2003, Proposition 2.2].

The following result shows other aspects of the synthesis operator:

Proposition 2.1 [Christensen and Stoeva 2003]. *Let $\{g_i\} \subseteq X^*$ be a p -frame. Then*

- (i) *the adjoint of T given by $T^* : X \rightarrow l^p; f \mapsto \{g_i(f)\}_{i=1}^{\infty}$ has closed range;*
- (ii) *X is reflexive;*
- (iii) *T is onto.*

The next proposition deals with preservation of the p -frame property under the action of various operators. Its proof is straightforward and we omit it.

Proposition 2.2. *Let X and Y be two Banach spaces and $\Psi : Y \rightarrow X$ be a bounded operator. Then*

- (i) *if $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p -Bessel sequence for X , then $\{\Psi^* g_i\}_{i=1}^{\infty}$ is a p -Bessel sequence for Y ;*
- (ii) *if $\{g_i\}_{i=1}^{\infty}$ is a p -frame for X , and Ψ is one-to-one with closed range, then $\{\Psi^* g_i\}_{i=1}^{\infty}$ is a p -frame for Y .*

Definition 2.3. Let X be a Banach space and $1 < p < \infty$. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is called a p -Riesz basis for X if the closed linear span of $\{f_i\}_{i=1}^{\infty}$ is X and there exist constants A and B such that, for any finite scalars $\{c_i\}$,

$$A\left(\sum |c_i|^p\right)^{\frac{1}{p}} \leq \left\| \sum c_i f_i \right\| \leq B\left(\sum |c_i|^p\right)^{\frac{1}{p}}. \tag{2-2}$$

Clearly, if $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p -Riesz basis for X^* then its synthesis operator has a bounded inverse. In particular, every p -Riesz basis for X^* is a q -frame for X with the same bounds.

If $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p -frame for X , then Proposition 2.1 shows that every $g \in X^*$ can be written as $g = \sum_{i=1}^{\infty} d_i g_i$ for some $\{d_i\}_{i=1}^{\infty} \in l^q$. Our aim is to find such a decomposition for the elements of X .

3. Main results

Let $\{f_i\}_{i=1}^{\infty}$ be a frame in a Hilbert space \mathcal{H} with the synthesis operator T . The canonical dual $\{S^{-1} f_i\}_{i=1}^{\infty}$ deals with the frame operator S ; see (1-2). It is not guaranteed that the canonical dual frame has the same structure as the frame itself

[Daubechies 1990]. Alternate duals are now presented as being a good candidate to apply the reconstruction formula (1-2).

Unfortunately, in p -frames, the frame operator cannot be defined. Hence, we first try to describe the canonical dual with respect to the synthesis operator. In fact, let $\{f_i\}_{i=1}^\infty$ be a frame in a Hilbert space \mathcal{H} with the analysis operator T^* . Then the frame condition (1-1) implies that T^* is injective and has closed range [Christensen 2008, Corollary 5.4.3]. Hence, the operator $(T^*)^{-1} : \mathcal{R}(T^*) \rightarrow \mathcal{H}$ can be extended to a bounded operator $\Phi : l^2 \rightarrow \mathcal{H}$. Therefore,

$$S^{-1} f_i = S^{-1} T \delta_i = S^{-1} T T^* \Phi \delta_i = \Phi \delta_i,$$

where $\{\delta_i\}_{i=1}^\infty$ is the canonical orthonormal basis for l^2 .

We summarize this fact in the following lemma.

Lemma 3.1. *Let $\{f_i\}_{i=1}^\infty$ be a frame in \mathcal{H} with the analysis operator T^* . The canonical dual $\{f_i\}_{i=1}^\infty$ can be represented as $\{\Phi \delta_i\}_{i=1}^\infty$, where $\Phi : l^2 \rightarrow \mathcal{H}$ is the unique extension of $(T^*)^{-1}$ and $\{\delta_i\}_{i=1}^\infty$ is the canonical orthonormal basis of l^2 .*

Let X be a Banach space with dual X^* and $1 < p < \infty$. The usual duality between X and X^* allows us to consider p -frames for X^* . In fact, a sequence $\{f_i\}_{i=1}^\infty \subseteq X$ is a p -frame if there exist constants A and B such that

$$A \|g\| \leq \left(\sum_{i=1}^{\infty} |g(f_i)|^p \right)^{\frac{1}{p}} \leq B \|g\| \quad (g \in X^*).$$

If the upper frame condition is satisfied we call $\{f_i\}_{i=1}^\infty$ a p -Bessel sequence.

Definition 3.2. Let $\{g_i\}_{i=1}^\infty \subseteq X^*$ be a p -Bessel sequence for X . A q -Bessel sequence $\{f_i\}_{i=1}^\infty \subseteq X$ for X^* is called a *dual* for $\{g_i\}_{i=1}^\infty$ if

$$g = \sum g(f_i) g_i \quad (g \in X^*) \quad \text{or} \quad f = \sum g_i(f) f_i \quad (f \in X). \quad (3-1)$$

If $\{g_i\}_{i=1}^\infty$ is p -frame, by using the Cauchy–Schwarz inequality, $\{f_i\}_{i=1}^\infty$ is automatically a q -frame for X^* , and vice versa. For more details see Theorem 2.10 of [Christensen and Stoeva 2003].

Denote the synthesis operators of $\{g_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^\infty$ by T and U , respectively. Also let X be reflexive. Then (3-1) holds if and only if $TU^* = I_{X^*}$ or $UT^* = I_X$. Although, for every $p \neq 2$, there exist a Banach space X and a p -frame for X without any dual [Casazza et al. 1999], Christensen and Stoeva [2003] showed that a p -frame $\{g_i\}_{i=1}^\infty$ has a dual if and only if $\mathcal{R}(T^*)$, the range of T^* , is complemented in l^p . Obviously, every p -Riesz basis for X^* has a unique dual.

Now we give a characterization of dual p -frames:

Proposition 3.3. *Let $\{g_i\}_{i=1}^\infty \subseteq X^*$ be a p -frame for X with the synthesis operator T . Then there exists a one-to-one correspondence between duals of $\{g_i\}_{i=1}^\infty$ and bounded left inverses of T^* .*

Proof. Suppose that $\Phi : l^p \rightarrow X$ is a bounded left inverse of T^* and consider $\{\delta_i\}_{i=1}^\infty$ as the canonical basis of l^p . It is obvious that $\{f_i\}_{i=1}^\infty := \{\Phi\delta_i\}_{i=1}^\infty$ is a q -Bessel sequence and

$$f = \Phi T^* f = \Phi \sum_{i=1}^{\infty} g_i(f)\delta_i = \sum_{i=1}^{\infty} g_i(f)f_i \quad (f \in X).$$

Thus $\{f_i\}_{i=1}^\infty$ is a q -frame for X^* . Conversely, let $\{f_i\}_{i=1}^\infty \subseteq X$ be a dual for $\{g_i\}_{i=1}^\infty$. Consider $\Phi : l^p \rightarrow X$ as the synthesis operator of $\{f_i\}_{i=1}^\infty$. Then Φ is bounded and for each $f \in X$ we have

$$f = \sum_{i=1}^{\infty} g_i(f)f_i = \sum_{i=1}^{\infty} g_i(f)\Phi\delta_i = \Phi(\{g_i(f)\}_{i=1}^\infty) = \Phi T^* f. \quad \square$$

As a consequence, we show that a p -frame $\{g_i\}_{i=1}^\infty \subseteq X^*$ with a unique dual is a q -Riesz basis for X^* . In fact, by [Proposition 3.3](#) there exists a one-to-one correspondence between the dual frames of $\{g_i\}_{i=1}^\infty$ and all bounded left inverse operators of T^* , in which T is the synthesis operator of $\{g_i\}_{i=1}^\infty$. Hence, $\{g_i\}_{i=1}^\infty$ has a unique dual if and only if T is injective. T is also surjective by [Proposition 2.1](#). Thus T is invertible and $\|T^{-1}\| < \infty$. This implies that $\{g_i\}_{i=1}^\infty$ is a q -Riesz basis for X^* .

Proposition 3.4. *Assume that p -frame $\{g_i\}_{i=1}^\infty \subseteq X^*$ has a dual. Then the q -Bessel sequence $\{f_i\}_{i=1}^\infty \subseteq X$ is a dual for $\{g_i\}_{i=1}^\infty$ if there exists a bounded operator $\Psi : X^* \rightarrow l^q$ such that $T\Psi = 0$. Conversely, all duals of $\{g_i\}_{i=1}^\infty$ (provided existence) can be described in this manner.*

Proof. Let $\{g_i\}_{i=1}^\infty$ be a p -frame. As a consequence of [Proposition 2.1](#), the operator $(T^*)^{-1} : \mathcal{R}(T^*) \rightarrow X$ is well-defined. If $\{g_i\}_{i=1}^\infty$ has a dual, then $\mathcal{R}(T^*)$ is complemented and so this operator can be extended to a bounded linear operator $W : l^p \rightarrow X$. Now assume that $\{f_i\}_{i=1}^\infty \subseteq X$ is a dual for $\{g_i\}_{i=1}^\infty$ with the synthesis operator U . Then (3-1) immediately implies that $TU^* = I$. Define $\Psi : X^* \rightarrow l^q$ by $\Psi = U^* - W^*$. Clearly, Ψ is a bounded operator and

$$T\Psi = TU^* - TW^* = I - (WT^*)^* = 0.$$

Conversely, suppose that $\Psi : X^* \rightarrow l^q$ is a bounded operator via $T\Psi = 0$. Take $\Phi = W - \psi^*$. Then Φ is a bounded operator and $\Phi T^* = I$. Using [Proposition 3.3](#) we conclude that $\{\Phi\delta_i\}_{i=1}^\infty$ is a dual for $\{g_i\}_{i=1}^\infty$. \square

Suppose that $\{g_i\}_{i=1}^\infty \subseteq X^*$ is a p -frame with the synthesis operator T such that $\mathcal{R}(T^*) \subseteq l^p$ is complemented. Then $\{W\delta_i\}_{i=1}^\infty$ is called the *canonical dual* of $\{g_i\}_{i=1}^\infty$, where $W : l^p \rightarrow X$ is the extension of $(T^*)^{-1}$. Other duals, which are characterized by [Proposition 3.3](#), are called *alternate duals*. In other words, the canonical dual is associated to the bounded inverse of T^* whereas alternate duals are in fact obtained by the left inverses of T^* .

Now we are ready to state a perturbation theorem about duals.

Theorem 3.5. *Let $\{g_i\}_{i=1}^\infty \subseteq X^*$ be a p -frame for X with bounds A_1 and B_1 . The p -frames sufficiently close to $\{g_i\}_{i=1}^\infty$ have a dual. More precisely, let $\{f_i\}_{i=1}^\infty \subseteq X$ be a dual for $\{g_i\}_{i=1}^\infty$ with bounds A_2 and B_2 , and let $\{g'_i\}_{i=1}^\infty$ be another p -frame with bounds A' and B' such that $\{g_i - g'_i\}_{i=1}^\infty$ is a p -Bessel sequence with constant sufficiently small ϵ . Then there exists a dual q -frame $\{f'_i\}_{i=1}^\infty$ for $\{g'_i\}_{i=1}^\infty$ such that $\{f_i - f'_i\}_{i=1}^\infty$ is also a q -Bessel sequence with bound multiplied by ϵ .*

Proof. Denote by T_1 and T_2 the synthesis operators of $\{g_i\}_{i=1}^\infty$ and $\{g'_i\}_{i=1}^\infty$, respectively. Then $\|T_1 - T_2\| < \epsilon$ by [Proposition 2.2](#) of [[Christensen and Stoeva 2003](#)]. Moreover, $(T_1^*)^{-1} : \mathcal{R}(T_1^*) \rightarrow X$ can be extended to a bounded operator $W : l^p \rightarrow X$ by the assumption. Hence

$$\|I - T_2^*W\| = \|(T_1 - T_2)^*W\| \leq \|W\|\|T_1 - T_2\| \leq \epsilon\|W\|.$$

Consequently T_2^*W is invertible. It follows that T_2^* has a bounded right inverse. A similar argument shows that its left inverse also exists. Consider $U_1 : l^p \rightarrow X$ as the synthesis operator of $\{f_i\}_{i=1}^\infty$. Then

$$\|I - U_1T_2^*\| = \|U_1(T_1 - T_2)^*\| \leq \|U_1\|\|T_1 - T_2\| \leq \epsilon\|U_1\|.$$

Therefore, the p -frame $\{g'_i\}_{i=1}^\infty$ has a dual. We are looking for the desired dual. First by [Proposition 3.4](#) there exists a bounded operator $\Psi : X^* \rightarrow l^q$ such that $T_1\Psi = 0$. Assume that $W_2 : l^p \rightarrow X$ is an extension of $(T_2^*)^{-1}$. Put

$$h_i = (W_2 + \Psi^*)\delta_i.$$

Then $\{h_i\}_{i=1}^\infty$ is a q -Bessel sequence with the synthesis operator $U_2 := W_2 + \Psi^*$ by [Proposition 2.2](#). Moreover, for each $f \in X$ we have

$$\begin{aligned} \|f - U_2T_2^*f\| &= \|\Psi^*T_2^*f\| \\ &= \|\Psi^*T_1^*f - \Psi^*T_2^*f\| \\ &\leq \|T_1 - T_2\|\|\Psi\|\|f\| \leq \epsilon\|\Psi\|\|f\|. \end{aligned}$$

Therefore, $U_2T_2^*$ is invertible for sufficiently small $\epsilon > 0$. In particular,

$$\|I - U_2T_2^*\| \leq \epsilon\|\Psi\|.$$

It remains to show that the q -frame $\{f'_i\}_{i=1}^\infty := \{(U_2 T_2^*)^{-1} f_i\}_{i=1}^\infty$ satisfies the theorem. It is easy to see that

$$U_2 T_2^* = I + \Psi^*(T_1^* - T_2^*). \quad (3-2)$$

Hence,

$$1 - \epsilon \|\Psi\| \leq \|U_2 T_2^*\|. \quad (3-3)$$

For each sequence $\{d_i\}_{i=1}^\infty$ in l^p by using (3-2) and (3-3) we get

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} d_i (f_i - f'_i) \right\| &= \|U_1 \{d_i\} - (U_2 T_2^*)^{-1} U_1 \{d_i\}\| \\ &\leq \|U_1\| \|I - (U_2 T_2^*)^{-1}\| \left(\sum_{i=1}^{\infty} |d_i|^p \right)^{\frac{1}{p}} \\ &\leq \|U_1\| \|(U_2 T_2^*)^{-1}\| \|U_2 T_2^* - I\| \left(\sum_{i=1}^{\infty} |d_i|^p \right)^{\frac{1}{p}} \\ &\leq \frac{\epsilon \|\Psi\| B_2}{1 - \epsilon \|\Psi\|} \left(\sum_{i=1}^{\infty} |d_i|^p \right)^{\frac{1}{p}}. \end{aligned}$$

This means that $\{f_i - f'_i\}_{i=1}^\infty$ is a q -Bessel sequence and its bound is a multiple of ϵ . \square

Let $\{g_i\}_{i=1}^\infty \subseteq X^*$ be a p -Bessel sequence for X with the synthesis operator T . We say that a q -Bessel sequence $\{f_i\}_{i=1}^\infty \subseteq X$ with the synthesis operator U is an *approximately dual* of $\{g_i\}_{i=1}^\infty$ if

$$\|I - T U^*\| < 1 \quad \text{or} \quad \|I - U T^*\| < 1. \quad (3-4)$$

Obviously, $\{f_i\}_{i=1}^\infty$ is a dual of $\{g_i\}_{i=1}^\infty$ when $T U^* = I$ or $U T^* = I$. Approximate duals are studied in a Hilbert space setting in [Christensen and Laugesen 2010]. They are easier to construct than the classical dual frames. For p -frames, which don't have duals in general, it is natural to ask whether we can exploit the approximate duals instead of duals. Unfortunately, the answer is negative. In fact, if $\{g_i\}_{i=1}^\infty$ is a p -frame for X with an approximate dual $\{f_i\}_{i=1}^\infty$, then, with notation as above, the operator $U T^*$ is invertible. Hence, $\{U T^* f_i\}_{i=1}^\infty$ is a p -frame by Proposition 2.2. Moreover,

$$f = (U T^*)^{-1} U T^* f = (U T^*)^{-1} \sum_{i=1}^{\infty} g_i(f) f_i = \sum_{i=1}^{\infty} g_i(f) (U T^*)^{-1} f_i.$$

Therefore, $\{(U T^*)^{-1} f_i\}$ is a dual of $\{g_i\}_{i=1}^\infty$.

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