involve
a journal of mathematics

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Audra McMillan

mathematical sciences publishers

2012 vol. 5, no. 1

INVOLVE 5:1(2012)

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(Communicated by John C. Wierman)

Holte introduced a $n \times n$ matrix P as a transition matrix related to the carries obtained when summing n numbers base b. Since then Diaconis and Fulman have further studied this matrix proving it to also be a transition matrix related to the process of b-riffle shuffling n cards. They also conjectured that the matrix P is totally nonnegative. In this paper, the matrix P is written as a product of a totally nonnegative matrix and an upper triangular matrix. The positivity of the leading principal minors for general n and b is proven as well as the nonnegativity of minors composed from initial columns and arbitrary rows.

1. Introduction

Holte [1997] introduced an $n \times n$ matrix P, with entries

$$P(i,j) = \frac{1}{b^n} \sum_{r=0}^{j-\lfloor i/b \rfloor} (-1)^r \binom{n+1}{r} \binom{n-1-i+(j+1-r)b}{n}$$

where the $P(i,j)$ entry gives the probability that when adding n random numbers base b, the next carry will be j, given that the previous carry was i. This matrix was then further studied in [2009a; Diaconis and Fulman 2009b], where it is noted that this is also a transition matrix related to card shuffling, where the $P(i,j)$ entry records the probability that a b-riffle shuffle of a permutation with i descents will lead to a permutation with j descents. Note that the rows and columns of this matrix are indexed by $0, \dots, n-1$.

Holte proved a number of properties of the matrix P, including that P has eigenvalues given by the geometric sequence $1, b^{-1}, \dots, b^{-(n-1)}$, implying that the determinant is positive for positive b.

A matrix will be referred to as *totally nonnegative* if every minor is nonnegative and *totally positive* if every minor is positive. Note that in some texts, such as [Pinkus 2010] and [Karlin 1968] these terms are replaced by *totally positive* and

MSC2010: 15B48, 60C05.

Keywords: total positivity, shuffle, minors.

strictly totally positive respectively. Totally nonnegative matrices figure prominently in a wide range of mathematical disciples including, but not limited to, combinatorics, stochastic processes and probability theory. Many properties of totally nonnegative matrices are known including eigenvalue/eigenvector properties and factorisation of such matrices. A good reference for the theory and applications of total positivity is [Pinkus 2010] and some further results on stochastic totally nonnegative matrices are included in [Gasca and Micchelli 1996].

Diaconis and Fulman [2009a, Remark after Lemma 4.2] conjectured that the matrix P is totally nonnegative for all positive integers n and b. Their paper included a proof that for all n and b, P is totally nonnegative of order 2, that is all the 2×2 minors are nonnegative, and that when b is a power of 2, P is totally nonnegative. Unfortunately, their method of proof does not generalise to other b. The aim of this paper is to make progress on the general conjecture.

Recall the following result:

Theorem 1. *Let $A = (a_{ij})$ be an $n \times n$ nonsingular matrix whose rows and columns are indexed by $0, \dots, n-1$. Then A is totally nonnegative if and only if A satisfies*

(i) $A\begin{pmatrix} 0, \dots, k-1 \\ 0, \dots, k-1 \end{pmatrix} > 0$ *for* $k = 1, \dots, n$,

(ii) $A\begin{pmatrix} i_1, \dots, i_k \\ 0, \dots, k-1 \end{pmatrix}$ *for* $0 \le i_1 < \cdots < i_k \le n-1$ *and* $k = 1, \dots, n$,

(iii) $A\begin{pmatrix} 0, \dots, k-1 \\ j_1, \dots, j_k \end{pmatrix}$ *for* $0 \le j_1 < \cdots < j_k \le n-1$ *and* $k = 1, \dots, n$,

where $A\begin{pmatrix} i_1, \dots, i_k \\ j_1, \dots, j_k \end{pmatrix}$ *denotes the minor composed of rows* $i_1, \dots, i_k$ *and columns* $j_1, \dots, j_k$.

A proof of this can be found in [Pinkus 2010, Proposition 2.15].

In this paper, we will prove (i) and (ii) for the matrix P, hence reducing the conjecture to condition (iii). Proving these conditions hold for P is equivalent to proving that they hold for $P' = b^n P$, so this matrix will be dealt with instead.

2. Proof of total nonnegativity claims

Firstly, note that

$$\begin{aligned} P'(i, j) &= \sum_{r=0}^{j-\lfloor i/b \rfloor} (-1)^r \binom{n+1}{r} \binom{n-1-i+(j+1-r)b}{n} \\ &= \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} \binom{n-1-i+(j+1-r)b}{n}, \end{aligned}$$

which implies

$$P' = \left(\binom{n-1-i+(j+1)b}{n}\right)_{\substack{0\le i\le n-1\\ 0\le j\le n-1}} \left((-1)^{j-i}\binom{n+1}{j-i}\right)_{\substack{0\le i\le n-1\\ 0\le j\le n-1}},$$

where $\binom{n}{k} = 0$ if $k < 0$. Let's call the first matrix A, and the second B. Note that B is upper unitriangular.

Using the Vandermonde convolution note that

$$\sum_{k=0}^{n-i} \binom{n-i}{n-i-k}\binom{jb}{i+k} = \binom{n-i+jb}{n},$$

so A can be further factored as

$$A = \left[\binom{n-i-1}{j-i}\right]_{i,j} \left[\binom{(j+1)b}{i+1}\right]_{i,j}.$$

Let's call these matrices C and D, respectively. Note that C is upper unitriangular, so this factorisation of P' implies that $\det P' = \det D$.

Lemma 2. *C is totally nonnegative.*

Proof. Obviously all the leading principal minors of C are 1, and all other minors composed of k initial columns and k arbitrary rows are 0 since C is upper unitriangular.

Now let $k \in \mathbb{Z}$, $1 \le k \le n$ and $0 \le j_1 < \cdots < j_k \le n-1$. Again using the Vandermonde convolution we observe that

$$\sum_{p=0}^{k-i-1} \binom{k-i-1}{p}\binom{n-k}{j_{l+1}-i-p} = \binom{n-i-1}{j_{l+1}-i},$$

so

$$C\binom{0,\dots,k-1}{j_1,\dots,j_k} = \left|\left[\binom{n-i-1}{j_{l+1}-i}\right]_{i,l}\right| = \left|\left[\binom{k-i-1}{l-i}\right]_{i,l}\right| \left|\left[\binom{n-k}{j_{l+1}-i}\right]_{i,l}\right|$$
$$= \left|\left[\binom{n-k}{j_{l+1}-i}\right]_{i,l}\right|. \qquad (*)$$

A sequence $(a_i)_{0\le i<\infty}$ is called a Pólya frequency sequence of infinite order if the corresponding infinite kernel matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

is totally nonnegative. The matrix (*) is nonnegative since it is a submatrix of the infinite kernel matrix of the sequence

$$\left(\binom{n-k}{0}, \binom{n-k}{1}, \dots, \binom{n-k}{n-k}\right),$$

which is a Pólya frequency sequence of infinite order according to the classification of Pólya frequency sequences in [Karlin 1968, Theorem 5.3, Chapter 8].

Therefore, by Theorem 1, matrix C is totally nonnegative for all n. □

Lemma 3. *D is totally nonnegative.*

Proof. D is a submatrix of the upper triangular Pascal matrix

$$\left[\binom{j}{i}\right]_{i,j}$$

which is simply the reflection of C about the antidiagonal where the dimension is $nb+1$, and hence is totally nonnegative [Pinkus 2010, Propositions 1.2 and 1.3]. Therefore D is totally nonnegative. □

Corollary 4. *A is totally nonnegative.*

Proof. Since the product of totally nonnegative matrices is totally nonnegative, A is totally nonnegative. □

Proposition 5. *Conditions* (*i*) *and* (*ii*) *of Theorem* 1 *hold for matrix P' for general n and b.*

Proof. Let $k \in \mathbb{Z}$, $1 \le k \le n$ and $0 \le i_1 < \cdots < i_k \le n-1$. From the Cauchy–Binet formula and the fact that B is upper unitriangular,

$$P'\begin{pmatrix} i_1, \dots, i_k \\ 0, \dots, k-1 \end{pmatrix} = A\begin{pmatrix} i_1, \dots, i_k \\ 0, \dots, k-1 \end{pmatrix} \ge 0$$

and

$$\begin{aligned} P'\begin{pmatrix} 0, \dots, k-1 \\ 0, \dots, k-1 \end{pmatrix} &= A\begin{pmatrix} 0, \dots, k-1 \\ 0, \dots, k-1 \end{pmatrix} \\ &= \sum_{0 \le m_1 < \cdots < m_k \le n-1} C\begin{pmatrix} 0, \dots, k-1 \\ m_1, \dots, m_k \end{pmatrix} D\begin{pmatrix} m_1, \dots, m_k \\ 0, \dots, k-1 \end{pmatrix} \\ &\ge D\begin{pmatrix} 0, \dots, k-1 \\ 0, \dots, k-1 \end{pmatrix} = \left| \left[\binom{(j+1)b}{i+1}\right]_{i,j} \right|. \end{aligned}$$

Here the inequality follows from the fact that C and D are totally nonnegative and C is upper unitriangular.

However this is simply the determinant of a smaller version of D, with n replaced by k and therefore by the previous factorisation of P', this is equal to the determinant of the P' matrix of dimension k, which is positive (as stated earlier) so we are done. □

One might hope that condition (iii) could be proved similarly by noting that

$$P'\binom{0,\ldots,k-1}{j_1,\ldots,j_k} = \sum_{0\le m_1<\cdots<m_k\le n-1} A\binom{0,\ldots,k-1}{m_1,\ldots,m_k} B\binom{m_1,\ldots,m_k}{j_1,\ldots,j_k}.$$

However the proof of condition (ii) relied on the fact that the minors of B involved were clearly seen to be 0 or 1 so this equation easily simplified. This is not the case for the above equation since little has been established about general minors of B. Progress might still be made if all minors of size k were nonnegative for some k however small examples show this to be unlikely, for example this is not true for minors of size 2 for any n. If the conjecture is true, it seems likely that a new approach is required to prove condition (iii).

3. Acknowledgements

This work was supported by the University of Sydney School of Mathematics vacation scholarship. A special thanks is given to Dr. Anthony Henderson for his support with this project.

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Received: 2011-02-24 Revised: 2011-07-17 Accepted: 2011-09-04

amcm7623@uni.sydney.edu.au *School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia*

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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Typeset in LaTeX

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2012 vol. 5 no. 1