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a journal of mathematics

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(Communicated by Joseph Gallian)

We consider the problem of describing the possible exponents of n -by- n boolean primitive circulant matrices. It is well known that this set is a subset of $[1, n-1]$ and not all integers in $[1, n-1]$ are attainable exponents. In the literature, some attention has been paid to the gaps in the set of exponents. The first three gaps have been proven, that is, the integers in the intervals $[\frac{n}{2}+1, n-2]$, $[\frac{n}{3}+2, \frac{n}{2}-2]$ and $[\frac{n}{4}+3, \frac{n}{3}-2]$ are not attainable exponents. Here we study the distribution of exponents in between those gaps by giving the exact exponents attained there by primitive circulant matrices. We also study the distribution of exponents in between the third gap and our conjectured fourth gap. It is interesting to point out that the exponents attained in between the $(i-1)$ -th and the i -th gap depend on the value of $n \bmod i$.

1. Introduction

A boolean matrix is a matrix over the binary boolean algebra $\{0, 1\}$. An n -by- n boolean matrix C is said to be circulant if each row of C (except the first one) is obtained from the preceding row by shifting the elements cyclically 1 column to the right. In other words, the entries of a circulant matrix $C = (c_{ij})$ are related in the manner: $c_{i+1,j} = c_{i,j-1}$, where $0 \leq i \leq n-2$, $0 \leq j \leq n-1$, and the subscripts are computed modulo n . The first row of C is called the generating vector. Here and throughout we number the rows and columns of an n -by- n matrix from 0 to $n-1$.

The set of all n -by- n boolean circulant matrices forms a multiplicative commutative semigroup C_n with $|C_n| = 2^n$ [Davis 1979; Lancaster 1969]. This semigroup

MSC2010: 05C25, 05C50, 11P70.

Keywords: exponent, primitive circulant matrix, basis of a cyclic group, order, box.

Bueno was supported by Dirección General de Investigación (Ministerio de Ciencia y Tecnología) of Spain under grant MTM2009-09281 and NSF grant DMS-0852065. Fang and Fuller were supported by NSF Grant DMS-0852065 for the REU program at the University of California, Santa Barbara. Furtado's contribution was done within the activities of Centro de Estruturas Lineares e Combinatórias da Universidade de Lisboa.

was thoroughly investigated by K. K.-H. Butler and J. R. Krabill [1974] and by S. Schwarz [1974].

An n -by- n boolean matrix C is said to be primitive if there exists a positive integer k such that $C^k = J$, where J is the n -by- n matrix whose entries are all ones and the product is computed in the algebra $\{0, 1\}$. The smallest such k is called the exponent of C , and we denote it by $\text{exp } C$. Let us denote $E_n = \{\text{exp } C : C \in C_n, C \text{ is primitive}\}$.

In [Bueno et al. 2009] we stated the following question: Given a positive integer n , what is the set E_n ?

The previous question can easily be restated in terms of circulant graphs or bases for finite cyclic groups, as we explain next.

Let C be a boolean primitive circulant matrix and let S be the set of positions corresponding to the nonzero entries in the generating vector of C (where the columns are counted starting with zero). C is the adjacency matrix of the circulant digraph $\text{Cay}(\mathbb{Z}_n, S)$. The vertex set of this graph is \mathbb{Z}_n and there is an arc from u to $u + a \pmod{n}$ for every $u \in \mathbb{Z}_n$ and every $a \in S$. A digraph D is called primitive if there exists a positive integer k such that for each ordered pair a, b of vertices there is a directed walk from a to b of length k in D . The smallest such integer k is called the exponent of the primitive digraph D . Thus, a circulant digraph G is primitive if and only if its adjacency matrix is. Moreover, if they are primitive, they have the same exponent. Therefore, finding the set E_n is equivalent to finding the possible exponents of circulant digraphs of order n .

Let n be a positive integer and let S be a nonempty subset of the additive group \mathbb{Z}_n . For a positive integer k we denote by kS the set given by

$$kS = \{s_1 + \dots + s_k \pmod{n} : s_i \in S\} \subset \mathbb{Z}_n.$$

The set kS is called the k -fold sumset of S .

The set S is said to be a basis for \mathbb{Z}_n if there exists a positive integer k such that $kS = \mathbb{Z}_n$. The smallest such k is called the order of S , denoted by $\text{order}(S)$. It is well known [Butler and Krabill 1974; Schwarz 1974] that the set $S = \{s_0, s_1, \dots, s_r\} \subset \mathbb{Z}_n$ is a basis if and only if $\text{gcd}(s_1 - s_0, \dots, s_r - s_0, n) = 1$. In [Bueno et al. 2009] we proved that, given a matrix C in C_n , if S is the set of positions corresponding to the nonzero entries in the generating vector of C , then C is primitive if and only if S is a basis for \mathbb{Z}_n . Moreover, if C is primitive, then $\text{exp}(C) = \text{order}(S)$. Therefore, finding the set E_n is equivalent to finding the possible orders of bases for the cyclic group \mathbb{Z}_n . This question is quite interesting by itself. We note that all the results in this paper will be given in terms of bases for \mathbb{Z}_n , as the techniques we can use following this approach result more convenient.

The problem we study in this paper has applications in different areas. In particular, circulant matrices appear as transition matrices in Markov processes

[Chou et al. 2008]. Also, the problem stated in terms of bases for \mathbb{Z}_n has applications in coding theory and quantum information [Klopsch and Lev 2009].

In the literature, the problem of computing all possible exponents attained by circulant primitive matrices or, equivalently, by circulant digraphs, has been considered. In particular, the following results were obtained. Here and throughout, $[a, b]$ denotes the set of positive integers in the real interval $[a, b]$. If $a > b$ then $[a, b] = \emptyset$.

Lemma 1 [Huang 1990; Wang and Meng 1997]. *If C is a primitive circulant matrix, then its exponent is either $n - 1$, $\lfloor \frac{n}{2} \rfloor$, $\lfloor \frac{n}{2} \rfloor - 1$ or does not exceed $\lfloor \frac{n}{3} \rfloor + 1$. Moreover, $\exp C = n - 1$ if and only if the number of nonzero entries in the generating vector of C is exactly 2.*

Lemma 2 [Dukes et al. 2010]. *For every $n \geq 3$, the sets $[\lfloor \frac{n}{4} \rfloor + 3, \lfloor \frac{n}{3} \rfloor - 2]$ and E_n are disjoint.*

All these results can be immediately translated into results about the possible orders of bases for a finite cyclic group.

Note that the only primitive matrix in C_2 is J_2 , so $E_2 = \{1\}$. From now on, we assume that $n \geq 3$. In [Bueno et al. 2009] we presented a conjecture concerning the possible exponents attained by n -by- n boolean primitive circulant matrices which we restate here in a more precise way. We start with a definition.

Definition. Let j be a positive integer. We call the j^{th} box of \mathbb{Z}_n , and denote it by B_j , the set of positive integers

$$\left[\left\lfloor \frac{n}{j} \right\rfloor - 1, \left\lfloor \frac{n}{j} \right\rfloor + j - 2 \right].$$

Conjecture 3. *If $C \in C_n$ is primitive, then*

$$\exp C \in \left[1, \lfloor \sqrt{n} \rfloor \right] \cup \bigcup_{j=1}^{\lfloor \sqrt{n} \rfloor} B_j.$$

In [Dukes et al. 2010], it was proven that if $C \in C_n$ is primitive and its exponent is greater than k for some positive integer k , then there exists d_k such that the exponent of C is within d_k of n/l for some integer $l \in [1, k]$. Notice that the result we present in Conjecture 3 produces gaps in the set of exponents which are larger than the ones encountered in [Dukes et al. 2010]. In fact, we have shown that the gaps in our conjecture should be maximal [Bueno and Furtado 2010]. We say that a gap A in E_n is maximal if $A' \cap E_n \neq \emptyset$ for any interval of integers $A' \subset [1, n - 1]$, with A strictly contained in A' . In [Bueno and Furtado 2010], we proved that for each positive integer j , there is an integer n , such that $B_{j,n}$ is a maximal gap in

E_n . However, as stated in [Dukes et al. 2010], we remain far from a complete characterization of the possible exponents of $n \times n$ primitive circulant matrices.

Lemmas 1 and 2 above show the gaps between the first and second box, between the second and third box, and between the third and fourth box when these boxes do not overlap. Here we present the distribution of orders of bases within the first three boxes by showing what orders are attained and which ones are not. The results for the first and second box were already known [Huang 1990; Wang and Meng 1997] and we include them for completeness. We also study the order of bases in the fourth box by giving orders that are attained and we conjecture that those are, in fact, the exact orders in that box. In addition, we also prove that all integers in $[1, \lfloor \sqrt{n} \rfloor]$ are attained by bases of \mathbb{Z}_n .

This paper is organized as follows. In Section 2 we state our main results and prove them in Section 4. In Section 3 we state and prove several auxiliary results concerning the order of bases for \mathbb{Z}_n , which will be used to prove our main theorems. The order of several bases for \mathbb{Z}_n with cardinality at most 4 that are relevant to our proofs is studied in the Appendix.

2. Main results

In this section, we give the exact orders attained by bases for \mathbb{Z}_n in the first three boxes of \mathbb{Z}_n . We also give orders attained in the fourth box. Notice that the results for the first and second box were already known [Huang 1990; Wang and Meng 1997] but we include them for completeness. Finally, we state that all integers up to $\lfloor \sqrt{n} \rfloor$ are in E_n .

The result for the first box is an immediate consequence of Lemma 1.

Theorem 4 [Huang 1990]. *For all n ,*

$$B_1 \subseteq E_n.$$

Concerning the second box, we have the following result obtained in [Huang 1990; Wang and Meng 1997]. In Section 4.1 we include a proof of it using the techniques for bases.

Theorem 5 [Huang 1990; Wang and Meng 1997]. *Let $n \geq 17$ be a positive integer.*

- *If n is even, then $B_2 \subseteq E_n$.*
- *If n is odd, then $B_2 \cap E_n = \lfloor \frac{n}{2} \rfloor$.*

The next two theorems are our main results and will be proven in Section 4. In our first result we assume a lower bound n_0 for n , which is the smallest value of n for which the theorem holds for all $n > n_0$. The possible orders in E_n , with $n < n_0$, appear in Tables 1 and 2. We observe that, for any n for which the box under study does not overlap with adjacent boxes, the theorem holds. We also notice that,

though we have a lower bound for n in our results, when $n \equiv 0 \pmod j$, $j = 3, 4$, B_j is a subset of E_n , for all n .

Theorem 6. *Let $n \geq 45$ be a positive integer.*

- *If $n \equiv 0 \pmod 3$, then $B_3 \subseteq E_n$.*
- *If $n \equiv 1 \pmod 3$, then $B_3 \cap E_n = \{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor\}$.*
- *If $n \equiv 2 \pmod 3$, then $B_3 \cap E_n = \{\lfloor \frac{n}{3} \rfloor + 1\}$.*

Theorem 7. *Let $n \geq 16$ be a positive integer.*

- *If $n \equiv 0 \pmod 4$, then $B_4 \subseteq E_n$.*
- *If $n \equiv 1 \pmod 4$, then $\{\lfloor \frac{n}{4} \rfloor + 2, \lfloor \frac{n}{4} \rfloor + 1, \lfloor \frac{n}{4} \rfloor\} \subseteq E_n$.*
- *If $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$, then $\{\lfloor \frac{n}{4} \rfloor + 2, \lfloor \frac{n}{4} \rfloor + 1\} \subseteq E_n$.*

Though we do not prove it, we conjecture that $\lfloor \frac{n}{4} \rfloor - 1 \notin E_n$ when $n \equiv 1 \pmod 4$ and $\lfloor \frac{n}{4} \rfloor - 1, \lfloor \frac{n}{4} \rfloor \notin E_n$ when $n \equiv 2, 3 \pmod 4$.

In Tables 1 and 2 we give the exact orders attained by bases for \mathbb{Z}_n with $n = 2, 3, 4, \dots, 104$. As the numerical experiments show, for each n there is a number of

n	E_n	n	E_n	n	E_n
2	1	23	1...8, 11, 22	44	1...13, 15, 21, 22, 43
3	1, 2	24	1...9, 11, 12, 23	45	1...16, 22, 44
4	1, 2, 3	25	1...9, 12, 24	46	1...13, 15, 16, 22, 23, 45
5	1, 2, 4	26	1...9, 12, 13, 25	47	1...13, 16, 23, 46
6	1, 2, 3, 5	27	1...10, 13, 26	48	1...17, 23, 24, 47
7	1, 2, 3, 6	28	1...10, 13, 14, 27	49	1...14, 16, 17, 24, 48
8	1...4, 7	29	1...10, 14, 28	50	1...14, 17, 24, 25, 49
9	1...4, 8	30	1...11, 14, 15, 29	51	1...14, 16, 17, 18, 25, 50
10	1...5, 9	31	1...11, 15, 30	52	1...15, 17, 18, 25, 26, 51
11	1...5, 10	32	1...11, 15, 16, 31	53	1...15, 18, 26, 52
12	1...6, 11	33	1...12, 16, 32	54	1...15, 17, 18, 19, 26, 27, 53
13	1...6, 12	34	1...12, 16, 17, 33	55	1...15, 18, 19, 27, 54
14	1...7, 13	35	1...10, 12, 17, 34	56	1...16, 19, 27, 28, 55
15	1...7, 14	36	1...13, 17, 18, 35	57	1...16, 18, 19, 20, 28, 56
16	1...8, 15	37	1...13, 18, 36	58	1...16, 19, 20, 28, 29, 57
17	1...6, 8, 16	38	1...11, 13, 18, 19, 37	59	1...16, 20, 29, 58
18	1...9, 17	39	1...14, 19, 38	60	1...17, 19, 20, 21, 29, 30, 59
19	1...7, 9, 18	40	1...14, 19, 20, 39	61	1...17, 20, 21, 30, 60
20	1...7, 9, 10, 19	41	1...12, 14, 20, 40	62	1...17, 21, 30, 31, 61
21	1...8, 10, 20	42	1...14, 15, 20, 21, 41	63	1...17, 20, 21, 22, 31, 62
22	1...8, 10, 11, 21	43	1...12, 14, 15, 21, 42	64	1...18, 21, 22, 31, 32, 63

Table 1. Orders of bases for \mathbb{Z}_n .

n E_n	n E_n
65 1 ... 14, 16, 17, 18, 22, 32, 64	85 1 ... 18, 20, 21, 22, 23, 28, 29, 42, 84
66 1 ... 18, 21, 22, 23, 32, 33, 65	86 1 ... 18, 20, 21, 22, 23, 29, 42, 43, 85
67 1 ... 18, 22, 23, 33, 66	87 1 ... 18, 20, 22, 23, 28, 29, 30, 43, 86
68 1 ... 19, 23, 33, 34, 67	88 1 ... 24, 29, 30, 43, 44, 87
69 1 ... 19, 22, 23, 24, 34, 68	89 1 ... 20, 22, 23, 24, 30, 44, 88
70 1 ... 15, 17, 18, 19, 23, 24, 34, 35, 69	90 1 ... 19, 21, 22, 23, 24, 29, 30, 31, 44, 45, 89
71 1 ... 19, 24, 35, 70	91 1 ... 21, 23, 24, 30, 31, 45, 90
72 1 ... 20, 23, 24, 25, 35, 36, 71	92 1 ... 19, 21, 22, 23, 24, 25, 31, 45, 46, 91
73 1 ... 20, 24, 25, 36, 72	93 1 ... 21, 23, 24, 25, 30, 31, 32, 46, 92
74 1 ... 20, 25, 36, 37, 73	94 1 ... 21, 23, 24, 25, 31, 32, 46, 47, 93
75 1 ... 16, 18, 19, 20, 24, 25, 26, 37, 74	95 1 ... 20, 22, 24, 25, 32, 47, 94
76 1 ... 21, 25, 26, 37, 38, 75	96 1 ... 26, 31, 32, 33, 47, 48, 95
77 1 ... 16, 18, 19, 20, 21, 26, 38, 76	97 1 ... 18, 20, 22, 24, 25, 26, 32, 33, 48, 96
78 1 ... 21, 25, 26, 27, 38, 39, 77	98 1 ... 22, 24, 25, 26, 33, 48, 49, 97
79 1 ... 18, 20, 21, 26, 27, 39, 78	99 1 ... 22, 25, 26, 32, 33, 34, 49, 98
80 1 ... 17, 19, 20, 21, 22, 27, 39, 40, 79	100 1 ... 21, 23, 24, 25, 26, 27, 33, 34, 49, 50, 99
81 1 ... 22, 26, 27, 28, 40, 80	101 1 ... 23, 25, 26, 27, 34, 50, 100
82 1 ... 17, 19, 20, 21, 22, 27, 28, 40, 41, 81	102 1 ... 21, 23, 25, 26, 27, 33, 34, 35, 50, 51, 101
83 1 ... 19, 21, 22, 28, 41, 82	103 1 ... 19, 21, 22, 23, 26, 27, 34, 35, 51, 102
84 1 ... 23, 27, 28, 29, 41, 42, 83	104 1 ... 19, 21, 22, 23, 25, 26, 27, 28, 35, 51, 52, 103

Table 2. Orders of bases for \mathbb{Z}_n .

consecutive orders that can be attained by bases of \mathbb{Z}_n . Though we prove [Theorem 8](#), according to our numerical experiments, we conjecture that at least all consecutive integers up to $2\sqrt{n} - 2$ are attainable orders.

Theorem 8. *Let n be a positive integer. Then $[1, \lfloor \sqrt{n} \rfloor] \subseteq E_n$.*

Though this result is cited in [\[Dukes et al. 2010\]](#), it seems that the paper where its proof is said to be is not available.

3. Order of bases for \mathbb{Z}_n

Computing the order of bases for \mathbb{Z}_n is, in general, a challenging task. In this section we introduce some results relative to the order of bases of \mathbb{Z}_n that will be helpful when proving our main results.

To start with, let us notice that the order of a basis S is invariant under shifts and multiplication by a unit of \mathbb{Z}_n , that is, for $a \in \mathbb{Z}_n$ and b a unit of \mathbb{Z}_n

$$\text{order}(S) = \text{order}(S+a), \quad \text{and} \quad \text{order}(S) = \text{order}(b*S) \tag{1}$$

where $b*S = \{bs \bmod n : s \in S\}$. In particular, this result implies that the set of orders attained by bases of \mathbb{Z}_n is the same as the set of orders attained by bases of \mathbb{Z}_n containing 0.

We now state some known results about the order of a basis for \mathbb{Z}_n . The following lemma gives an upper bound on the cardinality of a basis when a lower bound on its order is known.

Lemma 9 [Klopsch and Lev 2009]. *Let $n \in \mathbb{N}$ and $\rho \in [2, n - 1]$. Let S be a basis for \mathbb{Z}_n such that $\text{order}(S) \geq \rho$. Then*

$$|S| \leq \max \left\{ \frac{n}{d} \left(\left\lfloor \frac{d-2}{\rho-1} \right\rfloor + 1 \right) : d|n, d \geq \rho + 1 \right\}.$$

In particular, for each fixed $k \in \mathbb{N}$, if $\text{order}(S) \geq \frac{n}{k}$ and $n \gg 0$, then $|S| \leq 2k$.

The next lemma gives an upper and a lower bound on the order of some bases for \mathbb{Z}_n with cardinality 3.

Lemma 10 [Bueno and Furtado 2010]. *Let $2 \leq b \leq n - 1$. Then*

$$\left\lfloor \frac{n}{b} \right\rfloor \leq \text{order}(\{0, 1, b\}) \leq \left\lfloor \frac{n}{b} \right\rfloor + b - 2.$$

We now give the exact order of some particular bases for \mathbb{Z}_n that will be needed later. The next lemma shows, in particular, that the largest element of the j -th box, $j \leq \sqrt{n}$, belongs to E_n for all n .

Lemma 11 [Bueno et al. 2009]. *For $j \in \{1, 2, \dots, \lfloor \sqrt{n} \rfloor\}$,*

$$\text{order}(\{0, 1, j\}) = \left\lfloor \frac{n}{j} \right\rfloor + j - 2.$$

Lemma 12 [Bueno and Furtado 2010]. *Let $2 \leq j \leq \sqrt{n}$ be a positive integer. Then*

$$\text{order}(\{0, 1, \left\lfloor \frac{n}{j} \right\rfloor + 1\}) = \left\lfloor \frac{n}{j} \right\rfloor + j - 2.$$

Lemma 13 [Bueno et al. 2009]. *Let $2 \leq r \leq n - 1$ and $t = n - r \left\lfloor \frac{n}{r} \right\rfloor$. Then*

$$\text{order}(\{0, 1, 2, \dots, r - 1, r\}) = \begin{cases} \left\lfloor \frac{n}{r} \right\rfloor & \text{if } t \leq 1, \\ \left\lfloor \frac{n}{r} \right\rfloor + 1 & \text{if } t > 1. \end{cases}$$

Lemma 14. *Let $2 \leq r \leq n - 2$. Then*

$$\text{order}(\{0, 1, 2, \dots, r - 1, r + 1\}) = \left\lfloor \frac{n}{r + 1} \right\rfloor + 1.$$

Proof. Let $S = \{0, 1, 2, \dots, r - 1, r + 1\}$. It can be shown by induction on k that, for $k \geq 1$, $kS = [0, \dots, k(r + 1) - 2] \cup \{k(r + 1)\}$. Thus, $\text{order}(S) = k$ if and only if k is the minimum integer such that $k(r + 1) - 2 \geq n - 1$, which implies the result. \square

Lemma 15. *Suppose that m is a divisor of n and let $1 \leq q < m \leq n$. Then*

$$\text{order} \left(\bigcup_{i=0}^q (i + \langle m \rangle) \right) = \left\lceil \frac{m-1}{q} \right\rceil.$$

Proof. Let S be the basis in the statement. Note that $kS = \bigcup_{i=0}^{kq} (i + \langle m \rangle)$ for all $k \geq 1$. Therefore, the order of S equals the minimum k such that $kq \geq m - 1$ and the result follows. \square

As a consequence of the previous result, we obtain that, if j is a divisor of n , the smallest element of the j -th box is an element of E_n , since

$$\text{order} (\langle n/j \rangle \cup (1 + \langle n/j \rangle)) = n/j - 1.$$

Using canonical projections we can bound the order of some bases in a convenient way. Given \mathbb{Z}_n and a proper divisor m of n , we denote by ϕ the canonical quotient map $\phi : \mathbb{Z}_n \rightarrow \mathbb{Z}_{n/m}$. We denote by $\text{order}_n(S)$ the order of the basis S as a subset of \mathbb{Z}_n . The next result is well known. For that reason, we include it without proof.

Lemma 16. *Let m be a proper divisor of n . If S is a basis for \mathbb{Z}_n that contains zero and an element of order m , then $\phi(S)$ is a basis for $\mathbb{Z}_{n/m}$ and*

$$\text{order}_{n/m}(\phi(S)) \leq \text{order}_n(S) \leq \text{order}_{n/m}(\phi(S)) + m - 1.$$

The next corollaries are immediate consequences of the previous lemma and [Lemma 1](#).

Corollary 1 [[Huang 1990](#)]. *Suppose m is a proper divisor of n and S is a basis for \mathbb{Z}_n that contains zero and an element of order m . Then $\text{order}(S) \leq (n/m) + m - 2$.*

Corollary 2. *Let S be a basis for \mathbb{Z}_n and assume that S contains zero and an element of order 2. Then $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 1$ or $\text{order}(S) \geq \lfloor \frac{n}{2} \rfloor - 1$.*

Corollary 3. *Let S be a basis for \mathbb{Z}_n and assume that S contains zero and an element of order 3. Then $\text{order}(S) \leq \lfloor \frac{n}{6} \rfloor + 2$ or $\text{order}(S) \geq \lfloor \frac{n}{3} \rfloor - 1$.*

The next technical lemma allows us to prove [Corollary 4](#), which is a key result in the proof of our main theorems.

Lemma 17. *Let $j \geq 2$ be an integer and assume that*

$$b \in I_j = \left[\left\lfloor \frac{n}{j+1} \right\rfloor + 2, \left\lfloor \frac{n}{j} \right\rfloor - 1 \right].$$

Then

$$\text{order}(\{0, 1, b\}) \leq \left\lfloor \frac{n}{j+2} \right\rfloor + j.$$

Proof. Let $S = \{0, 1, b\}$. First we observe that $j + 1 < (j + 1)b - n < b$. We divide the proof into three cases.

Case 1: Assume b is even and $(j+1)b - n = b/2$. This implies that $(2j+1)b/2 = n$ and, therefore, b is not a divisor of n . Since $(2j+1)b = 2n$, then b is an element of \mathbb{Z}_n of order $2j+1$. Then

$$\text{order}(S) \leq \frac{n}{2j+1} + 2j - 1 \leq \left\lfloor \frac{n}{j+2} \right\rfloor + j.$$

The inequality on the left follows from [Corollary 1](#) while the right inequality follows after a few computations. Thus,

$$\left\lfloor \frac{n}{j+2} \right\rfloor + j = \left\lfloor \frac{(2j+1)(j+2+k)}{j+2} \right\rfloor + j \geq 3j+1+k = \frac{n}{2j+1} + 2j - 1.$$

Case 2: Assume $(j+1)b - n < b/2$. Let $k = j+1$ and $p = (j+1)b - n$. Clearly, $[0, k] \cup \{p\} \cup [b, b+k-1] \subseteq kS$. It can be shown by induction on q that

$$\bigcup_{i=0}^q [ip, ip+(q-i)k] \cup [b, b+qk-1] \subset qkS \quad (2)$$

and

$$\bigcup_{i=0}^{q-1} [ip+(k-1)b, ip+(k-1)b+(q-(i+1))k] \subset (qk-1)S. \quad (3)$$

Now assume that q is the largest integer such that $qp < b$, that is, $q = \lfloor b/p \rfloor$ and let $l = \max\{b-pq, p-k\}$. Note that $q \geq 2$. Also, the gaps between consecutive intervals in the unions in (2) and (3) have at most $l-1$ elements. Thus, we have

$$[0, b+j] \cup [jb, jb+(q-1)p+l] \subseteq (qk+l-1)S.$$

Moreover, $[0, jb+(q-1)p+l+j-1] \subseteq (qk+l-1+(j-1))S$. Since $n-jb = b-p$, we get that $(q-1)p+l+j \geq n-jb$ is equivalent to $l+j \geq b-pq$, which is true because of the definition of l . This implies

$$\text{order}(S) \leq qk + \max\{b-pq, p-k\} + j - 2. \quad (4)$$

Let $b = pq+r$, $0 \leq r < p$ and $q_1 = \lfloor rk/p \rfloor$. It is easy to show that

$$\max\{b-pq, p-k\} \leq q_1 + p - k \quad (5)$$

which implies

$$\text{order}(S) \leq \left\lfloor \frac{bk}{p} \right\rfloor + p - k + j - 2. \quad (6)$$

Taking into account (6), to complete the proof it is sufficient to show that

$$\left\lfloor \frac{bk}{p} \right\rfloor + p - k + j - 2 \leq \left\lfloor \frac{n}{j+2} \right\rfloor + j. \quad (7)$$

Let g be the function given by

$$g(b) = \frac{bk}{p} + p - 3 = \frac{n}{p} + p - 2.$$

To see that (7) holds it is enough to note that $g(b) \leq \frac{n}{j+2} + j$, or, equivalently,

$$b \in \left[\frac{n+j+2}{j+1}, \frac{n+\frac{n}{j+2}}{j+1} \right].$$

Case 3: Assume $(j+1)b - n > b/2$. Note that $j = \lfloor \frac{n}{b} \rfloor$. Let $n = jb + r_3, 0 \leq r_3 < b$. Thus, $(j+1)b - n = b - r_3$. Clearly, $[0, j+1] \cup \{b - r_3\} \cup [b, b+j] \cup [jb, jb+1] \subseteq (j+1)S$. It can be shown by induction on j that

$$[0, qj+1] \cup \bigcup_{i=0}^{q-1} [b - (q-i)r_3, b - (q-i)r_3 + ij] \cup [b, b+qj] \subset (qj+1)S \quad (8)$$

Denote by q the largest integer such that $qj+2 \leq b - qr_3$, that is, $q = \lfloor \frac{b-2}{j+r_3} \rfloor$. An argument similar to Case 2 implies that

$$\text{order}(S) \leq qj + \max\{r_3, b - q(j+r_3) - 1\} + j - 1. \quad (9)$$

Let $l = \max\{r_3, b - q(j+r_3) - 1\}$. Now we show that

$$qj+l+j-1 \leq \left\lfloor \frac{j(b-1)}{j+r_3} \right\rfloor + j+r_3 - 1 \leq \left\lfloor \frac{n}{j+2} \right\rfloor + j. \quad (10)$$

To see the first inequality in (10), it is enough to note that, by definition of q , $q(j+r_3) < b-1$ and $b-1 \leq (q+1)(j+r_3)$. To see the second inequality in (10), let h be the function given by

$$h(b) = \frac{j(b-1)}{j+r_3} + j+r_3 - 1 = \frac{n}{j+n-jb} + j+n-jb-2.$$

Then we see that

$$h(b) \leq \frac{n}{j+2} + (j+2) - 2 \text{ if and only if } j+n-jb \in \left[j+2, \frac{n}{j+2} \right].$$

Moreover, for $j+n-jb = \lfloor \frac{n}{j+2} \rfloor + 1$, we get $\lfloor h(b) \rfloor = \lfloor \frac{n}{j+2} \rfloor + j$, since by [Bueno and Furtado 2010](#), Theorem 5.7], and taking into account that $j < \sqrt{n}$,

$$\left\lfloor \frac{n}{\lfloor \frac{n}{j+2} \rfloor + 1} \right\rfloor = j+1.$$

Therefore, if $j+n-jb \in [j+2, \frac{n}{j+2}+1]$, or equivalently, if

$$b \in \left[\frac{n+j-1-\frac{n}{j+2}}{j}, \frac{n-2}{j} \right], \quad (11)$$

then the second inequality in (10) holds. We finish the proof by showing that any b satisfying our assumptions is such that (11) holds. Note that, as $(j+1)b-n > \frac{b}{2}$, we have $2n/(2j+1) < b \leq \lfloor \frac{n}{j} \rfloor - 1$. Thus, because $j \geq 2$, it follows that $b \leq \frac{n}{j} - 1 \leq \frac{n-2}{j}$. First we note that if $|I_j| \geq 2$, then

$$\frac{n+j-1-\frac{n}{j+2}}{j} \leq \frac{2n}{2j+1} < b. \quad (12)$$

If $|I_j| = 1$, then $b = \lfloor n/j \rfloor - 1$. If (12) holds, we are done. Otherwise, it can be proven that

$$\frac{n+j-1-\frac{n}{j+2}}{j} \leq b = \left\lfloor \frac{n}{j} \right\rfloor - 1. \quad \square$$

From the previous lemma we obtain the next corollary, which includes some results presented in [Dukes et al. 2010] without proof.

Corollary 4. *Let $n \geq 16$. Suppose that $2 \leq b \leq \lfloor \frac{n}{2} \rfloor + 1$.*

- (i) *If either $b \notin \{2, 3, \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1\}$, or $b = \lfloor \frac{n}{3} \rfloor$ and $n \not\equiv 0 \pmod{3}$, then $\text{order}(\{0, 1, b\}) \leq \lfloor \frac{n}{4} \rfloor + 2$.*
- (ii) *If either $b \in \{3, \lfloor \frac{n}{3} \rfloor + 1\}$, or $b = \lfloor \frac{n}{3} \rfloor$ and $n \equiv 0 \pmod{3}$, or $b = \lfloor \frac{n}{2} \rfloor$ with n odd, then $\text{order}(\{0, 1, b\}) = \lfloor \frac{n}{3} \rfloor + 1$.*
- (iii) *If either $b \in \{2, \lfloor \frac{n}{2} \rfloor + 1\}$, or $b = \lfloor \frac{n}{2} \rfloor$ and n is even, then $\text{order}(\{0, 1, b\}) = \lfloor \frac{n}{2} \rfloor$.*

Proof. By Lemma 17, if $b \in [\lfloor \frac{n}{4} \rfloor + 2, \lfloor \frac{n}{3} \rfloor - 1] \cup [\lfloor \frac{n}{3} \rfloor + 2, \lfloor \frac{n}{2} \rfloor - 1]$, the order of $\{0, 1, b\}$ is at most $\lfloor \frac{n}{4} \rfloor + 2$. By Lemma 10, if $4 \leq b \leq n/4$, then $\text{order}(\{0, 1, b\}) \leq \lfloor \frac{n}{4} \rfloor + 2$. By Lemma 12, $\text{order}(\{0, 1, \lfloor \frac{n}{4} \rfloor + 1\}) = \lfloor \frac{n}{4} \rfloor + 2$. If $b = \lfloor \frac{n}{3} \rfloor$ and $n \not\equiv 0 \pmod{3}$ then

$$\text{order}(\{0, 1, b\}) = \begin{cases} \text{order}(1+3*\{0, 1, b\}) = \text{order}(\{0, 1, 4\}) & \text{if } n \equiv 1 \pmod{3}, \\ \text{order}(2+3*\{0, 1, b\}) = \text{order}(\{0, 2, 5\}) & \text{if } n \equiv 2 \pmod{3}, \end{cases}$$

and the result follows from Lemmas 19 and 20. Thus, (i) follows. If $b \in \{3, \lfloor \frac{n}{3} \rfloor + 1\}$ the result follows from Lemmas 12 and 14. If n is odd, then

$$\text{order}(\{0, 1, \lfloor \frac{n}{2} \rfloor\}) = \text{order}(1+2*\{0, 1, b\}) = \{0, 1, 3\}$$

and the result follows from Lemma 14. If $n \equiv 0 \pmod{3}$ and $b = n/3$, then, for $k \geq 1$,

$$kS = [0, k] \cup [n/3, n/3+k-1] \cup [2n/3, 2n/3+k-2]$$

(in \mathbb{Z}). The order of S is the smallest positive integer k such that $k - 2 + 2n/3 \geq n - 1$, that is, $k = 1 + n/3$, which completes the proof of (ii). To prove (iii), note that, if n is even and $b = n/2$, then, for $k \geq 1$,

$$kS = [0, k] \cup [n/2, n/2 + k - 1]$$

(in \mathbb{Z}). Thus, the order of S is the smallest positive integer k such that $k - 1 + n/2 \geq n - 1$, that is, $\text{order}(S) = n/2$. If $b \in \{2, \lfloor \frac{n}{2} \rfloor + 1\}$, the result follows from Lemmas 12 and 13. \square

4. Proofs of the main results

In this section we prove Theorems 5, 6, 7, and 8. To prove the first three results, we initially show that certain orders in each box are attained by giving examples of bases with such orders. Then, regarding the first two theorems, we prove that the remaining orders are not attained.

4.1. Proof of Theorem 5. In the next table, we give examples of bases attaining the orders in the second box according to Theorem 5. The results follow from Lemmas 13 and 15.

Second Box for \mathbb{Z}_n		
$n \equiv 0 \pmod 2$	$n \equiv 1 \pmod 2$	Order(S)
$S = \langle n/2 \rangle \cup (1 + \langle n/2 \rangle)$	—	$\lfloor \frac{n}{2} \rfloor - 1$
$S = \{0, 1, 2\}$	$S = \{0, 1, 2\}$	$\lfloor \frac{n}{2} \rfloor$

We now assume that $n \geq 17$ and n is odd, and show that there is no basis $S \subseteq \mathbb{Z}_n$ such that $\text{order}(S) = \lfloor \frac{n}{2} \rfloor - 1$.

Assume that $S \subset \mathbb{Z}_n$ is a basis such that $\text{order}(S) = \lfloor \frac{n}{2} \rfloor - 1$. By Lemma 9, $|S| \leq 3$. Note that, by definition of basis, $|S| \geq 2$ and, by Lemma 1, $|S| \neq 2$ if $\text{order}(S) \neq n - 1$. Thus $|S| = 3$. Suppose $S = \{0, a, b\}$ where $a, b \in \mathbb{Z}_n$. If a had order $m \neq n$, then $3 \leq m < \lfloor \frac{n}{2} \rfloor$, since n is odd. By Corollary 1, this would imply that $\text{order}(S) \leq m + n/m - 2 < \lfloor \frac{n}{2} \rfloor - 1$, as $n \geq 17$. Therefore, a must have order n . Then S has the same order as $a^{-1}S = \{0, 1, c\}$ for some $c \in \mathbb{Z}_n$. If $c > \lfloor \frac{n}{2} \rfloor + 1$, then S has the same order as $1 - a^{-1}S = \{0, 1, d\}$ with $d \leq \lfloor \frac{n}{2} \rfloor + 1$. Thus, we can assume that $c \leq \lfloor \frac{n}{2} \rfloor + 1$. Now using Corollary 4, we get $\text{order}(S) \neq \lfloor \frac{n}{2} \rfloor - 1$, a contradiction.

4.2. Proof of Theorem 6. The next table gives examples of bases attaining the conjectured orders in the third box according to Theorem 6. The results follow from Lemmas 13–15.

Third Box for \mathbb{Z}_n			
$n \equiv 0 \pmod 3$	$n \equiv 1 \pmod 3$	$n \equiv 2 \pmod 3$	Order(S)
$S = \langle n/3 \rangle \cup (1 + \langle n/3 \rangle)$	—	—	$\lfloor \frac{n}{3} \rfloor - 1$
$S = \{0, 1, 2, 3\}$	$S = \{0, 1, 2, 3\}$	—	$\lfloor \frac{n}{3} \rfloor$
$S = \{0, 1, 3\}$	$S = \{0, 1, 3\}$	$S = \{0, 1, 3\}$	$\lfloor \frac{n}{3} \rfloor + 1$

The fact that, for $n \geq 45$, $\text{order}(S) \neq \lfloor \frac{n}{3} \rfloor - 1$, if $n \equiv 1 \pmod 3$, and $\text{order}(S) \notin \{ \lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{3} \rfloor \}$, if $n \equiv 2 \pmod 3$, follows from [Lemma 18](#). Just note that, if $\text{order}(S) \in \{ \lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{3} \rfloor \}$, then, by [Lemma 9](#), $|S| \leq 4$.

The statement of the next lemma is stronger than what is needed to prove [Theorem 6](#). However, the techniques we developed before allowed us to get this result, which in turn is useful in the proof of [Corollary 5](#).

Lemma 18. *Let $n \geq 45$ and suppose that 3 is not a divisor of n . Let S be a basis for \mathbb{Z}_n . If $|S| \leq 4$, then $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ or $\text{order}(S) \geq \lfloor \frac{n}{3} \rfloor$. Moreover, if $\text{order}(S) = \lfloor \frac{n}{3} \rfloor$, then $n \equiv 1 \pmod 3$.*

Proof. Without loss of generality, assume $0 \in S$. Suppose that $n \not\equiv 0 \pmod 3$. Since S is a basis, $|S| > 1$. If $|S| = 2$, then $\text{order}(S) = n - 1 > \lfloor \frac{n}{3} \rfloor$. Suppose that $|S| = 3$ or $|S| = 4$. If S has an element whose order is not 1, 2, $n/2$ nor n , then, by [Corollary 1](#), the result follows since there can't be an element of order 3 or $n/3$. Thus, this element has order ≥ 4 or $\leq n/4$. Suppose that the order of the elements in S is 1, 2, $n/2$, or n , where 2 and $n/2$ only occur when n is even. If S has an element of order 2, then the result follows from [Corollary 2](#). If S does not contain an element of order 2, then necessarily it contains an element of order n . Moreover, by (1), if S has an element of order n , the basis S has the same order as some basis of the form $\{0, 1, a, b\}$. If $|S| = 3$, then we can assume that $S = \{0, 1, a\}$, with $1 < a \leq \lfloor \frac{n}{2} \rfloor + 1$. In this case, the result follows from [Corollary 4](#). If $|S| = 4$, assume that $S = \{0, 1, a, b\}$ with $a \leq \lfloor \frac{n}{2} \rfloor + 1$. Since for $S' \subset S$, $\text{order}(S) \leq \text{order}(S')$, we have

$$\text{order}(\{0, 1, a, b\}) \leq \min\{\text{order}(\{0, 1, a\}), \text{order}(\{0, 1, b\})\}. \tag{13}$$

Let

$$A_1 = \{2, 3, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1\},$$

$$A_2 = \{2, 3, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, 1 - \lfloor \frac{n}{2} \rfloor, -\lfloor \frac{n}{3} \rfloor, -2, -1\}.$$

Note that $-\lfloor \frac{n}{2} \rfloor \in A_2$. Also, $1 - \lfloor \frac{n}{2} \rfloor \equiv \lfloor \frac{n}{2} \rfloor + 1 \pmod n$, for n even. If $a \notin A_1$ or $b \notin A_2$ then, by [Corollary 4](#) and taking into account (13),

$$\text{order}(\{0, 1, a, b\}) \leq \min\{\text{order}(\{0, 1, a\}), \text{order}(\{0, 1, b\})\} \leq \lfloor \frac{n}{4} \rfloor + 2.$$

Recall that $\text{order}(\{0, 1, 1-b\}) = \text{order}(\{0, 1, b\})$. If $a \in A_1$ and $b \in A_2$, the 25 and 26 \rightarrow result follows from Lemmas 22–26. \square

The following result was presented in [Dukes et al. 2010]. However, the authors leave most of the details of the proof to the reader and we do not see clearly that the result follows from their proof. For that reason and for completeness we are including it in this paper.

Corollary 5. *Let S be a basis for \mathbb{Z}_n . Then*

$$\text{order}(S) \notin \left[\left\lfloor \frac{n}{4} \right\rfloor + 3, \left\lfloor \frac{n}{3} \right\rfloor - 2 \right].$$

Proof. Note that, for $n < 45$, the interval in the statement is empty. Assume that $n \geq 45$. Without loss of generality, suppose that $0 \in S$. If $S \subset \mathbb{Z}_n$ is a basis such that $\left\lfloor \frac{n}{4} \right\rfloor + 3 \leq \text{order}(S)$, by Lemma 9, $|S| \leq 6$. Assume that $n \not\equiv 0 \pmod{3}$. If $|S| = 5$ or $|S| = 6$, by [Bueno et al. 2009, Theorem 3.7], $\text{order}(S) \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$. If $|S| \leq 4$, by Lemma 18, $\text{order}(S) \leq \left\lfloor \frac{n}{4} \right\rfloor + 2$ or $\text{order}(S) \geq \left\lfloor \frac{n}{3} \right\rfloor$.

Now assume that $n \equiv 0 \pmod{3}$. If $|S| = 3$, the result follows from Corollary 4. Suppose that $|S| \in \{4, 5, 6\}$. If $\left\lfloor \frac{n}{4} \right\rfloor + 3 \leq \text{order}(S)$, by Corollary 1, the order of the elements in S must be 1, 2, 3, $n/2$, $n/3$, or n . First note that S contains, or has the same order as a basis which contains, an element of order 2, 3 or n . In fact, if $|S| = 4$ and S does not have an element of order 2, 3 or n , then S has an element of order $n/2$ and an element of order $n/3$. Hence, $\{0, 2a, 3b\} \subseteq S$ for some $a, b \in \mathbb{Z}_n$. Since S is a basis, $3b - 2a$ is not an element of order $n/2$ nor $n/3$ as, otherwise, 6 would divide $2a$ or $3b$ and all elements of S would be multiples of 2 or multiples of 3. Thus, S has the same order as $S - 2a$, which has an element of order 2, 3 or n . A similar argument can be applied if $|S| = 5$ or $|S| = 6$. Thus, assume that S contains an element of order 2, 3 or n . If S contains an element of order 2 or 3, the result follows from Corollaries 2 and 3. Now suppose that S contains an element of order n and no elements of order 2 and 3. If either $n/3 + 1 \in S$ or n is even and $n/2 + 1 \in S$, then S can be transformed into a basis with the same order containing zero and an element of order 2 or 3 and we reduce the problem to the previous case. Let

$$\begin{aligned} A_1 &= \left\{ 2, 3, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1 \right\}, \\ A_2 &= \left\{ 2, 3, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1, 1 - \left\lfloor \frac{n}{2} \right\rfloor, -2, -1 \right\}. \end{aligned}$$

Assume that $S = \{0, 1, a, b, c, d\}$, with $a \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $b = c = d$ if $|S| = 4$, and $c = d$ if $|S| = 5$. Note that if $S' \subset S$ then $\text{order}(S) \leq \text{order}(S')$. If $a \notin A_1$ or, $b, c, d \notin A_2$ the result follows from Corollary 4. Suppose that $a \in A_1, b, c, d \in A_2$ and if a, b, c or $d \in \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1, 1 - \left\lfloor \frac{n}{2} \right\rfloor \right\}$ then n is odd. If $|S| = 4$, the result follows from Lemmas 22–26. If $|S| = 5$ or $|S| = 6$ the result follows from the Remark on page 204 by noting that S has a subset of cardinality 4 containing 0 and 1 which is not one of the exceptional bases and, therefore, $\text{order}(S) \leq \left\lfloor \frac{n}{4} \right\rfloor + 2$. \square

4.3. Proof of Theorem 7. The next table gives examples of bases attaining the orders in the fourth box of \mathbb{Z}_n claimed in Theorem 7. The results follow from Lemmas 12–15.

Fourth Box for \mathbb{Z}_n				
$n \equiv 0 \pmod 4$	$n \equiv 1 \pmod 4$	$n \equiv 2 \pmod 4$	$n \equiv 3 \pmod 4$	Order(S)
$\langle n/4 \rangle \cup (1 + \langle n/4 \rangle)$	—	—	—	$\lfloor \frac{n}{4} \rfloor - 1$
$\{0, 1, 2, 3, 4\}$	$\{0, 1, 2, 3, 4\}$	$\bigcup_{i=0}^2 (i + \langle n/2 \rangle)$	—	$\lfloor \frac{n}{4} \rfloor$
$\{0, 1, 2, 4\}$	$\{0, 1, 2, 4\}$	$\{0, 1, 2, 4\}$	$\{0, 1, 2, 4\}$	$\lfloor \frac{n}{4} \rfloor + 1$
$\{0, 1, \frac{n}{4} + 1\}$	$\{0, 1, \lfloor \frac{n}{4} \rfloor + 1\}$	$\{0, 1, \lfloor \frac{n}{4} \rfloor + 1\}$	$\{0, 1, \lfloor \frac{n}{4} \rfloor + 1\}$	$\lfloor \frac{n}{4} \rfloor + 2$

4.4. Proof of Theorem 8. If $n \leq 4$, the result follows from Table 1. Assume $n \geq 5$. Notice that \mathbb{Z}_n is always a basis for \mathbb{Z}_n , which implies that $1 \in E_n$. Consider the set $S = \{0, 1, 2, \dots, r - 1, r + 1\}$ with $2 \leq r \leq n - 2$. By Lemma 14, $\text{order}(S) = \lceil \frac{n+1}{r+1} \rceil$. For all $r \geq \sqrt{n} - 1$

$$\frac{n+1}{r+1} - \frac{n+1}{r+2} = \frac{n+1}{(r+1)(r+2)} = \frac{n+1}{r^2+3r+2} \leq \frac{n+1}{n+\sqrt{n}} < 1.$$

It can be easily seen that, for positive real numbers a and b , $\lceil a \rceil - \lceil b \rceil \leq \lceil a - b \rceil$. Thus, $\lceil \frac{n+1}{r+1} \rceil - \lceil \frac{n+1}{r+2} \rceil \leq 1$ for all $r \geq \sqrt{n} - 1$, which implies that all integers from 2 to

$$\left\lceil \frac{n+1}{\lfloor \sqrt{n} \rfloor - 1 + 1} \right\rceil$$

are attained orders. But $\left\lceil \frac{n+1}{\lfloor \sqrt{n} \rfloor} \right\rceil \geq \left\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rceil \geq \lfloor \sqrt{n} \rfloor$ and the result follows.

Appendix: Gallery of bases and their orders

Here we provide the order of some particular bases that are necessary to prove the main results in this paper. We do not include all the proofs since many of them are similar.

Lemma 19. For $n \geq 6$, $\text{order}(\{0, 1, 4\}) = \lfloor \frac{n}{4} \rfloor + 2$.

Proof. Let $S = \{0, 1, 4\}$. It can be shown by induction on k that in \mathbb{Z} , for all $k \geq 2$,

$$kS = [0, 4k - 6] \cup [4k - 4, 4k - 3] \cup \{4k\}.$$

Let $q = \lfloor \frac{n}{4} \rfloor$. Then

$$(q + 1)S = [0, 4q - 2] \cup [4q, 4q + 1] \cup \{4q + 4\}$$

and $[0, 4q + 2] \subseteq (q + 2)S$. Note that $4q + 4 \not\equiv 4q - 1 \pmod n$, since $n \geq 6$. Thus, $(q + 1)S \not\equiv \mathbb{Z}_n \pmod n$. On the other hand, $4q + 2 \geq n - 1$. The result follows. \square

Lemma 20. For $n \geq 6$, $\text{order}(\{0, 2, 5\}) \leq \lfloor \frac{n}{5} \rfloor + 3$.

Lemma 21. For $n \geq 4$, $\text{order}(\{0, 2, 3, 4\}) = \lfloor \frac{n}{4} \rfloor + 1$.

Bases of the form $\{0, 1, 2, a\}$

Lemma 22. Let $n \geq 21$. Let $a \in \{3, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, 1 - \lfloor \frac{n}{2} \rfloor, -\lfloor \frac{n}{3} \rfloor, -2, -1\}$ and $S = \{0, 1, 2, a\}$. Then $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ or $\text{order}(S) \geq \lfloor \frac{n}{3} \rfloor - 1$. Moreover, if $n \equiv 1 \pmod{3}$, then $\text{order}(S) \neq \lfloor \frac{n}{3} \rfloor - 1$ and if $n \equiv 2 \pmod{3}$, then $\text{order}(S) \notin \{\lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{3} \rfloor\}$.

Proof. Case 1: If $a \in \{3, -1\}$, then the basis S has the same order as $\{0, 1, 2, 3\}$ and the result follows by [Lemma 13](#).

Case 2: If $a = -2$, then S has the same order as $2 + S = \{0, 2, 3, 4\}$ and the result follows from [Lemma 21](#).

Case 3: Suppose that $a \in \{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, 1 - \lfloor \frac{n}{2} \rfloor\}$. Assume n is even. Note that $1 - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$. In this case, S contains an element of order 2 or it has the same order as a basis containing 0 and an element of order 2. Thus, the result follows from [Corollary 2](#). Assume n is odd. Then

$$\begin{aligned} \text{order}(\{0, 1, 2, \lfloor \frac{n}{2} \rfloor\}) &= \text{order}(\{0, 1, 3, 5\}) \leq \text{order}(\{0, 1, 5\}), \\ \text{order}(\{0, 1, 2, \lfloor \frac{n}{2} \rfloor + 1\}) &= \text{order}(\{0, 1, 2, 4\}) \leq \text{order}(\{0, 1, 4\}). \end{aligned}$$

In both cases, $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ by [Corollary 4](#). Also,

$$\text{order}(\{0, 1, 2, \lfloor \frac{n}{2} \rfloor + 2\}) = \text{order}(\{0, 2, 3, 4\}) \leq \lfloor \frac{n}{4} \rfloor + 2$$

by [Lemma 21](#). Note that $1 - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + 2$.

Case 4: Suppose that $a \in \{-\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + 1\}$. If $n \equiv 0 \pmod{3}$, then S contains an element of order 3 or it has the same order as a basis containing 0 and an element of order 3. Thus, the result follows from [Corollary 3](#). Let $n \equiv 1 \pmod{3}$. If $a = -\lfloor \frac{n}{3} \rfloor$, then $3 * S = \{0, 1, 3, 6\}$ and

$$\text{order}(S) = \text{order}(3 * S) \leq \text{order}(\{0, 1, 6\});$$

if $a = \lfloor \frac{n}{3} \rfloor + 1$, then $3 * S - 2 = \{0, 1, 4, -2\}$ and

$$\text{order}(S) = \text{order}(3 * S - 2) \leq \text{order}(\{0, 1, 4\}).$$

In both cases, $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ by [Corollary 4](#). If $n \equiv 2 \pmod{3}$, then

$$\begin{aligned} \text{order}(\{0, 1, 2, -\lfloor \frac{n}{3} \rfloor\}) &= \text{order}(\{0, 1, 4, -2\}), \\ \text{order}(\{0, 1, 2, \lfloor \frac{n}{3} \rfloor + 1\}) &= \text{order}(\{0, 1, 3, 6\}), \end{aligned}$$

and the result follows as before. \square

Bases of the form $\{0, 1, 3, a\}$

Lemma 23. Let $n \geq 30$. Let $a \in \{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, 1 - \lfloor \frac{n}{2} \rfloor, -\lfloor \frac{n}{3} \rfloor, -2, -1\}$ and $S = \{0, 1, 3, a\}$. Then $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ or $\text{order}(S) \geq \lfloor \frac{n}{3} \rfloor - 1$. Moreover, if $n \equiv 1 \pmod{3}$, then $\text{order}(S) \neq \lfloor \frac{n}{3} \rfloor - 1$ and if $n \equiv 2 \pmod{3}$, then $\text{order}(S) \notin \{\lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{3} \rfloor\}$.

Bases of the form $\{0, 1, \lfloor \frac{n}{3} \rfloor + 1, a\}$

Lemma 24. Let $n \geq 30$. Let $a \in \{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, 1 - \lfloor \frac{n}{2} \rfloor, -\lfloor \frac{n}{3} \rfloor, -2, -1\}$ and $S = \{0, 1, \lfloor \frac{n}{3} \rfloor + 1, a\}$. Then $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ or $\text{order}(S) \geq \lfloor \frac{n}{3} \rfloor - 1$. Moreover, if $n \equiv 1 \pmod{3}$, then $\text{order}(S) \neq \lfloor \frac{n}{3} \rfloor - 1$ and if $n \equiv 2 \pmod{3}$, then $\text{order}(S) \notin \{\lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{3} \rfloor\}$.

Bases of the form $\{0, 1, \lfloor \frac{n}{2} \rfloor, a\}$

Lemma 25. Let $n \geq 22$. Let $a \in \{\lfloor \frac{n}{2} \rfloor + 1, 1 - \lfloor \frac{n}{2} \rfloor, -\lfloor \frac{n}{3} \rfloor, -2, -1\}$ and $S = \{0, 1, \lfloor \frac{n}{2} \rfloor, a\}$. Then $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ or $\text{order}(S) \geq \lfloor \frac{n}{3} \rfloor - 1$. Moreover, if $n \equiv 1 \pmod{3}$, then $\text{order}(S) \neq \lfloor \frac{n}{3} \rfloor - 1$ and if $n \equiv 2 \pmod{3}$, then $\text{order}(S) \notin \{\lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{3} \rfloor\}$.

Proof. If n is even, then S contains an element of order 2 and the result follows from [Corollary 2](#). Now suppose that n is odd. Note that $2 * S + 1 = \{0, 1, 3, 2a + 1\}$.

For $a = -1$, $\text{order}(S) = \text{order}(\{0, 1, 3, -1\}) = \text{order}(\{0, 1, 2, 4\}) \leq \lfloor \frac{n}{4} \rfloor + 2$, by [Corollary 4](#).

For $a = -\lfloor \frac{n}{3} \rfloor$ and $n \equiv 0 \pmod{3}$, S contains an element of order 3 and the result follows from [Corollary 4](#).

For $a = \lfloor \frac{n}{2} \rfloor + 1$, $\text{order}(S) = \text{order}(2 * S + 1) = \{0, 1, 2, 3\}$ and the result follows from [Lemma 13](#).

Now suppose that a does not satisfy the previous cases. We have $\text{order}(S) = \text{order}(\{0, 1, 3, b\})$, with $b \in \{4, \lfloor \frac{n}{3} \rfloor + t + 1, -3\}$, where $0 < t = n - 3\lfloor \frac{n}{3} \rfloor \leq 2$. Thus, $\text{order}(S) \leq \text{order}(\{0, 1, b\}) \leq \lfloor \frac{n}{4} \rfloor + 2$ by [Corollary 4](#). \square

Bases of the form $\{0, 1, \lfloor \frac{n}{2} \rfloor + 1, a\}$

Lemma 26. Let $n \geq 21$, $a \in \{1 - \lfloor \frac{n}{2} \rfloor, -\lfloor \frac{n}{3} \rfloor, -2, -1\}$ and $S = \{0, 1, \lfloor \frac{n}{2} \rfloor + 1, a\}$. Then $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ or $\text{order}(S) \geq \lfloor \frac{n}{3} \rfloor - 1$. Moreover, if $n \equiv 1 \pmod{3}$, then $\text{order}(S) \neq \lfloor \frac{n}{3} \rfloor - 1$ and if $n \equiv 2 \pmod{3}$, then $\text{order}(S) \notin \{\lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{3} \rfloor\}$.

Proof. If n is even, then S has the same order as $S - 1$, which contains 0 and an element of order 2. Thus, the result follows from [Corollary 2](#). Now suppose that n is odd. Then $\text{order}(S) = \text{order}(\{0, 1, 2, 2a\})$.

If $a = 1 - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + 2$, then $2a = 3$ and the result follows from [Lemma 13](#).

If $a = -2$, then $2a = -4$ and, by [Corollary 4](#),

$$\text{order}(S) \leq \text{order}(\{0, 1, -4\}) = \text{order}(\{0, 1, 5\}) \leq \lfloor \frac{n}{4} \rfloor + 2.$$

If $a = -1$, then $2a = -2$ and, by [Lemma 21](#), $\text{order}(S) = \text{order}(\{0, 2, 3, 4\}) \leq \lfloor \frac{n}{4} \rfloor + 2$.

Suppose that $a = -\lfloor \frac{n}{3} \rfloor$. If $n \equiv 0 \pmod{3}$, then S contains 0 and an element of order 3 and the result follows from [Corollary 3](#). If $n \equiv 1 \pmod{3}$, then $S = \{0, 1, 2, \lfloor \frac{n}{3} \rfloor + 1\}$ and the result follows from [Lemma 22](#). If $n \equiv 2 \pmod{3}$, then, by [Corollary 4](#),

$$\text{order}(S) = \text{order}(\{0, 1, 2, \lfloor \frac{n}{3} \rfloor + 2\}) \leq \text{order}(\{0, 1, \lfloor \frac{n}{3} \rfloor + 2\}) \leq \lfloor \frac{n}{4} \rfloor + 2. \quad \square$$

Remark. Suppose that $S = \{0, 1, a, b\}$, with $a \in \{2, 3, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1\}$ and $b \in \{2, 3, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, 1 - \lfloor \frac{n}{2} \rfloor, -2, -1\}$, where n is odd if a or b belong to the set $\{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, -\lfloor \frac{n}{2} \rfloor\}$. From the proofs of [Lemmas 22–26](#), we get that $\text{order}(S) \leq \lfloor \frac{n}{4} \rfloor + 2$ if S is not one of the next exceptional bases:

$$\{0, 1, 2, 3\}, \quad \{0, 1, 2, -1\}, \quad \{0, 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1\}, \quad \{0, 1, \lfloor \frac{n}{2} \rfloor + 1, 1 - \lfloor \frac{n}{2} \rfloor\}.$$

Note that all of them have the same order as $\{0, 1, 2, 3\}$.

Acknowledgements

We would like to thank the anonymous referee whose comments and suggestions helped us improve the presentation of our paper.

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Received: 2011-06-10

Revised: 2011-09-21

Accepted: 2011-09-22

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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Typeset in L^AT_EX

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2012

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