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We consider numerical semigroups $\mathbb{N} \cap \mathbb{N}I$, for intervals I. We compute the Frobenius number and multiplicity of such semigroups, and show that we may freely restrict I to be open, closed, or half-open, as we prefer.

Given an interval $I \subseteq \mathbb{Q}^{>0}$ in the positive rationals, consider the set

$$S(I) = \mathbb{N} \cap \mathbb{N}I = \{m \in \mathbb{N} : \exists\, n \in \mathbb{N}, m/n \in I\}.$$

This turns out to be a numerical semigroup, and has been the subject of considerable recent investigation (see [Rosales and García-Sánchez 2009, Chapter 4] for an introduction). Special cases include modular numerical semigroups [Rosales et al. 2005], where $I = [m/n, m/(n-1)]$ $(m, n \in \mathbb{N})$; proportionally modular numerical semigroups [Rosales et al. 2003], where $I = [m/n, m/(n-s)]$ $(m, n, s \in \mathbb{N})$; and opened modular numerical semigroups [Rosales and Urbano-Blanco 2006] where $I = (m/n, m/(n-1))$ $(m, n \in \mathbb{N})$.

We consider instead arbitrary open intervals $I = (a, b)$. We show that this set of semigroups coincides with the set of semigroups generated by closed and half-open intervals. Consequently, this class of semigroups contains modular numerical semigroups, proportionally modular numerical semigroups, as well as opened modular numerical semigroups. We also compute two important invariants of these numerical semigroups: the Frobenius number $g(S(I))$ and multiplicity $m(S(I))$.

1. Preliminaries

We begin by defining a helpful function $\kappa(a, b)$. For $a, b \in \mathbb{R}$ with $a < b$ we define $\kappa(a, b) = \lfloor b/(b-a) \rfloor$. The function κ has various nice properties, for example $\kappa(a, b) = \kappa(ac, bc)$ for $c > 0$. In the special case of $a = m/n, b = m/(n-s)$, we have $\kappa(a, b) = \lfloor n/s \rfloor$. The following properties of $\kappa(a, b)$ are needed in the sequel.

Lemma 1.1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $b \neq 0$. Set $\kappa = \kappa(a, b)$. If $\kappa \neq 0$, then $(\kappa - 1)/\kappa \leq a/b$. If $\kappa \neq -1$, then $a/b < \kappa/(\kappa + 1)$.*

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Proof. We have $\kappa \le b/(b-a) < \kappa+1$. Assume that $\kappa \notin \{-1, 0\}$. Then

$$\kappa, \frac{b}{b-a}, \kappa+1$$

all have the same sign, and we have $1/\kappa \ge (b-a)/b > 1/(\kappa+1)$, hence

$$\frac{1-\kappa}{\kappa} \ge -\frac{a}{b} > \frac{-\kappa}{\kappa+1},$$

and the results follow. If $\kappa = 0$, then $0 \le b/(b-a) < 1$, so $b > 0$ and $b < b-a$ so $a < 0$ hence $a/b < 0 = \kappa/(\kappa+1)$. If $\kappa = -1$, then $-1 \le b/(b-a) < 0$, so $b < 0$ and $a - b \le b$ so $a \le 2b$ and $a/b \ge 2 = (\kappa-1)/\kappa$. □

Lemma 1.2. *Let $a, b \in \mathbb{R}$ with $a < b$ and $b > 0$. Then*

$$\mathbb{N} \setminus S((a,b)) = \mathbb{N} \cap \bigcup_{n=1}^{\kappa(a,b)} [b(n-1), an].$$

Proof. Because $S((a,b)) = \mathbb{N} \cap \cup_{n=1}^{\infty}(an, bn)$, we have

$$\mathbb{N} \setminus S((a,b)) = \mathbb{N} \cap \bigcup_{n=1}^{\infty} [b(n-1), an].$$

Since $b > 0$, $\kappa(a,b) \ne -1$ and hence by Lemma 1.1, $b\kappa(a,b) > a(\kappa(a,b)+1)$. Hence for $n > \kappa(a,b)$, the intervals are empty and may be excluded. □

Lemma 1.2 yields an upper bound for g. This bound will later be improved in Theorem 3.1, but for the purposes of Theorem 2.3 the following weaker bound suffices.

Corollary 1.3. *Suppose $0 < a < b$. Then $g(S((a,b))) \le \lfloor a\kappa(a,b) \rfloor$.*

2. Intervals

We now prove that restricting I to be open is harmless, as this class of semigroups coincides with ones generated by closed or half-open intervals. To reduce the number of cases to consider, we introduce the symbols $\{, \}$ to denote endpoints of an interval that are either open or closed. For example, $(a, b\}$ indicates an interval that is open on the left. The meaning of these symbols is determined when first used, and then remains consistent; that is, if $(a, b\}$ is open then $[a', b\}$ means $[a', b)$, and if $(a, b\}$ is half-open then $[a', b\}$ means $[a', b]$.

The following lemma is the cornerstone of the interval equivalence results. Let $d(x)$ denote the denominator of reduced rational x.

Lemma 2.1. *Let $a \in \mathbb{Q}^{>0}$, $n \in \mathbb{N}$. Then all rationals in the interval*

$$\left(a - \frac{a}{nd(a)+1}, a + \frac{a}{nd(a)+1}\right),$$

other than possibly a, have numerator greater than n.

Proof. Suppose $a = p/q$, so $d(a) = q$. Consider any rational x/y with

$$0 < \left|\frac{x}{y} - a\right| < \frac{a}{nq+1} = \frac{p}{q(nq+1)}.$$

Also, we have

$$\left|\frac{x}{y} - \frac{p}{q}\right| = \left|\frac{xq - yp}{yq}\right| \geq \frac{1}{yq},$$

because $xq - yp \neq 0$ since $x/y \neq a$. Combining, we get

$$\frac{p}{q(nq+1)} > \frac{1}{yq},$$

hence

$$\frac{px}{nq+1} > \frac{x}{y} > a - \frac{a}{nq+1} = \frac{pnq}{q(nq+1)} = \frac{pn}{nq+1},$$

and thus $x > n$. $\square$

Lemma 2.2. *Suppose that I, J, $I \cup J$ are all intervals. Then $S(I \cup J) = S(I) \cup S(J)$. Also, if $I \subseteq J$, then $S(I) \subseteq S(J)$ and $g(S(I)) \geq g(S(J))$.*

Proof. An integer $m \in S(I \cup J)$ if and only if $m/n \in I \cup J$ for some n. This is true if and only if $m/n \in I$ or $m/n \in J$. Hence $m \in S(I \cup J)$ if and only if $m \in S(I) \cup S(J)$. If $I \subseteq J$, then $S(J) = S(I \cup J) = S(I) \cup S(J) \supseteq S(I)$. $\square$

The following theorem lets us replace a closed endpoint with an open one nearby, leaving the semigroup unchanged. Given a modular or proportionally modular numerical semigroup S, it explicitly gives an open interval I with $S(I) = S$.

Theorem 2.3. *Let $0 < a < b$. Set*

$$a' = \begin{cases} a - \dfrac{a}{\lfloor a\kappa(a,b)\rfloor d(a)+1} & \text{if } a \in \mathbb{Q}, \\ a & \text{if } a \notin \mathbb{Q}, \end{cases}$$

$$b' = \begin{cases} b + \dfrac{b}{\lfloor a\kappa(a,b)\rfloor d(b)+1} & \text{if } b \in \mathbb{Q}, \\ b & \text{if } b \notin \mathbb{Q}. \end{cases}$$

Then

$$S([a,b\}) = S((a',b\}) \quad \text{and} \quad S(\{a,b]) = S(\{a,b')).$$

Proof. We consider only $[a,b\}$; $\{a,b]$ is symmetric. Suppose first that $a \notin \mathbb{Q}$. By Lemma 2.2, $S([a',b\}) = S((a',b\}) \cup S([a',a']) = S((a',b\})$ since $S([a',a']) = \varnothing$. We now assume $a \in \mathbb{Q}$. Since $a < 2a - a'$, Lemma 2.2 implies that $S((a',b\}) = S((a',2a-a')) \cup S([a,b\})$. We will show $S((a',2a-a')) \subseteq S([a,b\})$, implying $S((a',b\}) \subseteq S([a,b\})$ (and $S((a',b\}) \supseteq S([a,b\})$ by Lemma 2.2).

Let $c \in S((a',2a-a'))$. Hence there is some $d \in \mathbb{N}$ so that $c/d \in (a',2a-a')$. By Lemma 2.1, either $c/d = a$ (in which case $c \in S([a,b\})$), or $c > \lfloor a\kappa(a,b)\rfloor$.

In the latter case, we apply Corollary 1.3 and $c > \lfloor a\kappa(a,b)\rfloor \geq g(S((a,b))) \geq g(S([a,b\})))$, so $c \in S([a,b\})$. $\square$

Theorem 2.4 is a counterpoint to Theorem 2.3, allowing us to replace an open endpoint with a closed one nearby. Proposition 5 in [Rosales and Urbano-Blanco 2006] tells us more: that every $S(I)$ is proportionally modular; that is, there are $m, n, s \in \mathbb{N}$ where $S(I) = S([m/n, m/(n-s)])$. Unfortunately neither of these results give an explicit formula such as in Theorem 2.3.

Theorem 2.4. *Let $0 < a < b$. Then there are a', b' with $S((a,b\}) = S([a',b\})$ and $S(\{a,b)) = S(\{a',b])$. Further, $a/a', b/b' \in \mathbb{Q}$.*

Proof. We consider only $(a,b\}$; $\{a,b)$ is symmetric. Suppose first that $a \notin \mathbb{Q}$. By Lemma 2.2, $S([a',b\}) = S((a',b\}) \cup S([a',a']) = S((a',b\})$ since $S([a',a']) = \varnothing$. We now assume $a \in \mathbb{Q}$. Let a_0 be any rational in (a,b), and consider the sequence given by $a_i = \frac{1}{2}(a + a_{i-1})$, for $i \geq 1$. By Lemma 2.2, we have $S([a_1,b\}) \subseteq S([a_2,b\}) \subseteq \cdots \subseteq S((a,b\})$. Set

$$X = S((a,b\}) \setminus S([a_1,b\}),$$

a finite set. Set $Z = \{x/y : x \in X, x/y \in (a,b\}\}$. Since $a_i \to a$ and $\min Z > a$, there is some $j > 0$ with $Z \subseteq [a_j,b\}$, and hence $X \subseteq S([a_j,b\})$. We take $a' = a_j$; note that $a' \in \mathbb{Q}$ by construction. $\square$

3. Calculating $g(S((a,b)))$ and $m(S((a,b)))$

We now improve on Corollary 1.3 by calculating $g(S((a,b)))$ exactly. Various other results are known in related contexts. For example, if $S([a,b])$ is not a half-line, in [Rosales and Urbano-Blanco 2006] it was shown that

$$\frac{g(S([a,b]))}{g(S([a,b]))-1} < a < b < g(S([a,b])).$$

Also, if $2 \leq a < b$ with $a, b \in \mathbb{N}$, in [Rosales and Vasco 2009] it was shown that $g(S((a,b))) = b$.

Theorem 3.1. *Suppose $0 < a < b$. Set $\kappa = \kappa(a,b)$, $\kappa' = max(\kappa(a-1,b-1),0)$. Then $g(S((a,b))) = \lfloor a\alpha\rfloor$, where $\alpha \in \mathbb{Z}$ satisfies $\kappa' \leq \alpha \leq \kappa$. Specifically,*

$$\alpha = \kappa - \sum_{i=\kappa'+1}^{\kappa} \prod_{j=i}^{\kappa} \left(1 + \lfloor aj\rfloor + \lfloor b(1-j)\rfloor\right).$$

Proof. By Lemma 1.2, $g(S((a,b))) = \lfloor a\alpha\rfloor$, for the greatest integer α where

$$\mathbb{N} \cap [b(\alpha-1), a\alpha]$$

is nonempty; in particular $\alpha \le \kappa$. The lower bound $\alpha \ge \kappa'$ is trivial when $\kappa' = 0$; if $b \le 1$ then

$$\kappa(a-1, b-1) = \left\lfloor \frac{b-1}{b-a} \right\rfloor \le 0,$$

and hence $\kappa' = 0$. Otherwise, $b > 1$ and so by Lemma 1.1,

$$\frac{\kappa'-1}{\kappa'} \le \frac{a-1}{b-1};$$

rearranging we get $b(\kappa'-1) \le a\kappa' - 1$. Hence the interval $[b(\kappa'-1), a\kappa']$ has length at least 1. It must therefore contain an integer, so $\alpha \ge \kappa'$.

To prove the formula for α, for $i \le \kappa$ we define the function

$$f(i) = \begin{cases} 1 & \text{if } \alpha \le i, \\ 0 & \text{if } \alpha > i; \end{cases}$$

this gives us $\alpha = \kappa - \sum_{i=0}^{\kappa} f(i) = \kappa - \sum_{i=\kappa'+1}^{\kappa} f(i)$. We define f via $f(i) = \prod_{j=i}^{\kappa} \chi(j)$, for

$$\chi(j) = \begin{cases} 1 & \text{if } [b(j-1), aj] \cap \mathbb{N} \ne \varnothing, \\ 0 & \text{if } [b(j-1), aj] \cap \mathbb{N} = \varnothing. \end{cases}$$

We now have $\alpha = \kappa - \sum_{i=\kappa'+1}^{\kappa} \prod_{j=i}^{\kappa} \chi(j)$.

We now calculate $\chi(j)$ explicitly by showing that for $j \ge \kappa' + 1$, the interval $[b(j-1), aj]$ contains at most one integer. For $b \le 1$, we have $bj > aj \ge aj + (b-1)$ so $b(j-1) > aj - 1$. For $b > 1$, by Lemma 1.1 we have

$$\frac{a-1}{b-1} < \frac{\kappa'}{\kappa'+1} \le \frac{j-1}{(j-1)+1}, \qquad \text{for any } j-1 \ge \kappa'.$$

Rearranging, we get $b(j-1) > aj - 1$. Hence $\left|[b(j-1), aj] \cap \mathbb{N}\right| \le 1$ and in fact $\chi(j)$ equals the number of integers in $[b(j-1), aj]$, that is,

$$\chi(j) = 1 + \lfloor aj \rfloor + \lfloor b(1-j) \rfloor. \qquad \square$$

We have $\alpha \in [\kappa', \kappa]$; in general, neither bound can be improved. The size of this interval, $\kappa - \kappa'$, can be arbitrarily large, when b/a is small. On the other hand, the following shows that $\kappa - \kappa'$ is small if $b/a > 2$. This is desirable, as it shortens the calculation for $g(S(I))$.

Proposition 3.2. *Let $0 < 2a < b$. Let κ, κ' be as in Theorem 3.1. Then*

$$\kappa - \kappa' = \begin{cases} 1 & \text{if } a < 1, \\ 0 & \text{if } a \ge 1. \end{cases}$$

Proof. For convenience, set

$$I = \Big(\frac{b-1}{b-a}, \frac{b}{b-a}\Big);$$

$\kappa - \kappa'$ counts the number of integers in I. Suppose first that $b \leq 1$. Then

$$\kappa(a-1, b-1) \leq 0,$$

so $\kappa' = 0$. Note that $b > 2a$ implies $b - a > a$, and hence $1/(b-a) < 1/a$, so $1 + a/(b-a) < 1 + a/a = 2$, and hence $\kappa = \lfloor 1 + a/(b-a) \rfloor = 1$. Suppose now that $a < 1 < b$. If $a \leq \frac{1}{2}$, then $b > 1 = \frac{1}{2} + a$. Alternatively, if $a > \frac{1}{2}$, then $b > 2a > \frac{1}{2} + a$. Hence $b > \frac{1}{2} + a$; rearranging we get $1/(b-a) < 2$. Hence I is of length less than 2, and can contain at most one integer. Therefore $\kappa - \kappa' \leq 1$. But I contains the integer $1 = (b-a)/(b-a)$, so $\kappa - \kappa' = 1$. Lastly, we consider the case $a \geq 1$. We have $b - 1 \geq b - a$, hence $(b-1)/(b-a) \geq 1$ and I does not contain 0 or 1. Suppose I contains integer $n \geq 2$. Then $2 \leq b/(b-a)$; rearranging we get $b \leq 2a$, a contradiction. Hence I contains no integers, and $\kappa - \kappa' = 0$. □

Computing $m(S((a, b)))$ is similar to computing $g(S((a, b)))$, in that we must count integers in an interval, only this time the intervals are open. We first prove a technical lemma, for which we recall Farey sequences (for an introduction see [Graham et al. 1994]). The nth Farey sequence F_n consists of all reduced fractions in [0, 1] whose denominator is at most n, arranged in increasing order. The key property we require is that if a/b are c/d are consecutive terms in a Farey sequence, then $bc - ad = 1$.

Lemma 3.3. *Let* $0 < a < b$. *Let* $n \in \mathbb{N}$ *be minimal such that* (an, bn) *contains an integer. Suppose* $n > 1$. *Then* (an, bn) *contains exactly one integer.*

Proof. Suppose by way of contradiction that (an, bn) contains at least two integers. Then there is some $m \in \mathbb{N}$ such that $m, m+1 \in (an, bn)$. Set $d = \gcd(m, n)$. If $d > 1$ then $m/d \in (an/d, bn/d)$ violates the minimality of n. Similarly, $\gcd(m+1, n) = 1$. Let $m' \in (0, n-1)$ with $m = m' + kn$ for some integer k. We now consider the nth Farey sequence F_n. Both m'/n and $(m'+1)/n$ are elements of F_n; however $(m'+1)n - m'n = n > 1$, so they are not consecutive terms and there must be some p/q in F_n with $m'/n < p/q < (m'+1)/n$, with $q < n$. But then

$$\frac{m' + kn}{n} < \frac{p + qk}{q} < \frac{m' + 1 + kn}{n},$$

so $p + qk \in (aq, bq)$, violating the minimality of n. □

We now compute the multiplicity $m(S((a, b)))$. The reverse problem of finding an open interval whose semigroup possesses a given multiplicity, is solved in [Rosales and Vasco 2009]. A nondiscrete version is proved as Proposition 5 in [Rosales et al. 2003].

Theorem 3.4. *Suppose* $0 < a < b$. *Set* $\kappa'' = \kappa(1, b-a+1)$. *Then*

$$m(S((a,b))) = \lceil a\alpha \rceil,$$

where $\alpha \in \mathbb{N}$ *satisfies* $1 \leq \alpha \leq \kappa''$. *Specifically,*

$$\alpha = \sum_{i=0}^{\kappa''} \prod_{j=1}^{i} (2 + \lfloor aj \rfloor + \lfloor -bj \rfloor).$$

Proof. Set $m = m(S((a,b)))$, and let $\alpha \in \mathbb{N}$ be minimal such that $m/\alpha \in (a,b)$; then $m(S((a,b))) = \lceil a\alpha \rceil$. By Lemma 1.1, $1/(b-a+1) < \kappa''/(\kappa''+1)$. Rearranging, we find $\kappa'' b - \kappa'' a > 1$, so there is an integer $t \in (\kappa'' a, \kappa'' b)$. Suppose that $\alpha > \kappa''$. We then have $m/\alpha < m/\kappa'' \leq t/\kappa''$; since m/α and t/κ'' are in (a,b), we conclude that $m/\kappa'' \in (a,b)$, which contradicts the minimality of α. Hence $\alpha \leq \kappa''$.

We now prove the α formula. We proceed in a manner similar to Theorem 3.1, by defining

$$f(i) = \begin{cases} 1 & i \leq \alpha, \\ 0 & i > \alpha, \end{cases}$$

via $f(i) = \prod_{j=1}^{i}(1 - \chi(j))$, where $\chi(j)$ is the number of integers in (aj, bj). For $i < \alpha$, $\chi(i) = 0$. By Lemma 3.3, $\chi(\alpha) = 1$, so $f(i) = 0$ for $i \geq \alpha$. Hence

$$\alpha = \sum_{i=0}^{\kappa''} f(i) = \sum_{i=0}^{\kappa''} \prod_{j=1}^{i} (1 - \chi(j)),$$

but $1 - \chi(j) = 2 + \lfloor aj \rfloor + \lfloor -bj \rfloor$. □

We have $\alpha \in [1, \kappa'']$; in general, neither bound can be improved. The upper bound κ'' can be arbitrarily large, when $b - a$ is small. On the other hand, the following shows that κ'' is small if $b - a$ is large, thus simplifying computation of m.

Proposition 3.5. *Let* $0 < a < b$. *Let* $n \in \mathbb{N}$ *be minimal with* $b - a > 1/n$. *Then* $\kappa'' = n$, *in the notation of Theorem 3.4.*

Proof. We have $1/n < b - a \leq 1/(n-1)$, hence $n > 1/(b-a) \geq n-1$, so $\lfloor 1/(b-a) \rfloor = n - 1$, and

$$\kappa'' = \lfloor \frac{b-a+1}{b-a} \rfloor = \lfloor 1 + \frac{1}{b-a} \rfloor = 1 + (n-1) = n. \qquad \square$$

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