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The most general planar transformations that map parabolas into parabolas

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Consider the space of vertical parabolas in the plane interpreted generally to include nonvertical lines. It is proved that an injective map from a closed region bounded by one such parabola into the plane that maps vertical parabolas to other vertical parabolas must be the composition of a Laguerre transformation with a nonisotropic dilation. Here, a Laguerre transformation refers to a linear fractional or antilinear fractional transformation of the underlying dual plane.

1. Introduction

A familiar result in complex analysis is that in the extended complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the Möbius transformations map circles and lines to other circles and lines. In a beautiful paper from 1937, reprinted in [Blair 2000], Carathéodory proved the following converse result.

Theorem 1 ([Carathéodory 1937]). *Every arbitrary one-to-one correspondence between the points of a circular disc C and a bounded point set C' by which circles lying completely in C are transformed into circles lying in C' must always be either a direct or inverse transformation of Möbius.*

So not only are the circle preserving maps of $\widehat{\mathbb{C}}$ the Möbius transformations, but even locally, these are the only transformations that can map circles to circles.

In this paper we consider the analogous problem for the extended dual plane, $\widehat{\mathbb{D}} = \mathbb{D} \cup L_\infty$, where

$$\mathbb{D} = \{z = x + jy : x, y \in \mathbb{R},\ j^2 = 0\} \quad \text{and} \quad L_\infty = \{(\alpha j)^{-1} : \alpha \in \mathbb{R}\}.$$

Here, the linear fractional transformations are the Laguerre transformations. They map vertical parabolas and nonvertical lines to other vertical parabolas and nonvertical lines. We prove the following.

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Theorem 2. *Every injective map from a closed region bounded by a vertical parabola or nonvertical line that maps vertical parabolas and nonvertical lines to vertical parabolas and nonvertical lines is the composition of a nonisotropic dilation* $d_\lambda : (x, y) \to (\lambda x, \lambda^2 y)$, $0 \neq \lambda \in \mathbb{R}$, *with a direct or indirect Laguerre transformation.*

This theorem arose from work that two of the authors did to solve a Beckman–Quarles-type theorem for the dual plane. In particular, they proved:

Theorem 3 ([Ferdinands and Kavlie 2009]). *Suppose T is a bijective mapping on the space of vertical parabolas so that for vertical parabolas A, B,*

$$\delta(A, B) = 1 \text{ if and only if } \delta(T(A), T(B)) = 1.$$

Then T is induced by a Laguerre transformation of the dual plane.

Here, δ is the distance between (intersecting) parabolas and is measured as the difference in slopes at their point of intersection. An important step in the proof of this theorem uses a simpler version of Theorem 2 that has a very different proof.

We mention that planar Laguerre geometry often refers to a geometry of oriented circles where distance is measured as the length of the common tangent. The connection to dual numbers is made clear in Yaglom [1968]. We mention, too, that the transformations we call Laguerre transformations can also be interpreted as parabolic Möbius transformations. See, for instance, the recent survey by Kisil [2007].

2. Geometry in the extended dual plane

Here we summarize the properties and geometry of the dual numbers. A comprehensive account is given by Yaglom [1968].

A dual number $z \in \mathbb{D}$ is a formal expression $z = x + jy$ where $x, y \in \mathbb{R}$ and $j^2 = 0$. These numbers form a commutative algebra over $\mathbb{R}$ where addition and multiplication are done in the obvious way. One identifies dual numbers with points in the real plane via $x + jy \in \mathbb{D} \leftrightarrow (x, y) \in \mathbb{R}^2$, just as in the case of complex numbers. The coordinates of $z = x + jy$ are the real part and dual part, respectively. So $x = \text{Real}\,(x + jy)$ and $y = \text{Dual}\,(x + jy)$. Figure 1 illustrates the geometry of the dual plane. In particular, addition in $\mathbb{D}$ is done by adding position vectors. Multiplication is done by multiplying the real parts and adding the slopes of the position vectors. Because of this, the modulus and argument of z are usually defined by $|z| \overset{\text{def}}{=} |x|$ and $\arg z \overset{\text{def}}{=} y/x$.

The direct and indirect Laguerre transformations are the linear fractional and antilinear fractional transformations,

$$\mu(z) = \frac{az + b}{cz + d} \quad \text{and} \quad \mu(z) = \frac{a\bar{z} + b}{c\bar{z} + d},$$

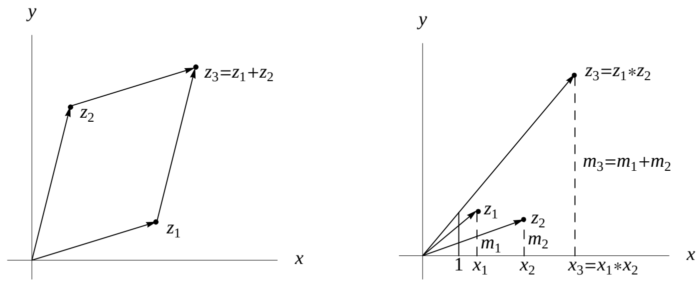

Figure 1. Addition and multiplication of dual numbers.

respectively, where $a, b, c, d \in \mathbb{D}$ and $ad - bc = 1$. The condition $ad - bc = 1$ acts as a normalization and has no effect on the transformation itself. It is just necessary that Real $(ad - bc) \neq 0$, or else μ maps $\widehat{\mathbb{D}}$ to a line or point.

The direct Laguerre transformations form a group that is isomorphic to $SL_2(\mathbb{D})$. Similar to Möbius transformations, they are generated by the following types:

(i) [translation] : $\mu(z) = z + b$ for $b \in \mathbb{D}$;

(ii) [rotation and isotropic dilation] : $\mu(z) = az$ for $a \in \mathbb{D}$, Real $(a) \neq 0$;

(iii) [inversion] : $\mu(z) = 1/z$.

These transformations preserve angles (measured as differences in slope); the indirect Laguerre transformations reverse angles.

Both the direct and indirect transformations preserve the space of vertical parabolas and nonvertical lines. (By a vertical parabola we mean that the axis of symmetry is vertical. The vertical parabolas and nonvertical lines can be described collectively as the graphs $y = rx^2 + sx + t$ for $r, s, t \in \mathbb{R}$.) This fact can be verified for the direct transformations by considering the three kinds of motions mentioned above. It then follows for indirect transformations, too, since obviously the conjugation $z \to \overline{z} = x - jy$ preserves the space.

By using stereographic projection, the extended dual plane $\widehat{\mathbb{D}} = \mathbb{D} \cup L_\infty$ can be viewed as an infinite cylinder as shown in Figure 2. (Laguerre transformations that do not arise as translations or similarities correspond with affine symmetries of the cylinder.) In this model, the set $L_\infty = \{(\alpha j)^{-1} : \alpha \in \mathbb{R}\}$ corresponds with a line of points at infinity. By using the transformation $\mu(z) = 1/z$ one can see that the parabola $y = rx^2 + sx + t$ intersects L_∞ at the point $-1/(rj)$, where $r, s, t \in \mathbb{R}$. In particular, nonvertical lines are the parabolas that intersect L_∞ at $1/(0j)$.

3. Proof of Theorem 2

The proof is modeled on Carathéodory's [Blair 2000]. We begin with an injective transformation $T : D \to D'$ that maps vertical parabolas and nonvertical lines

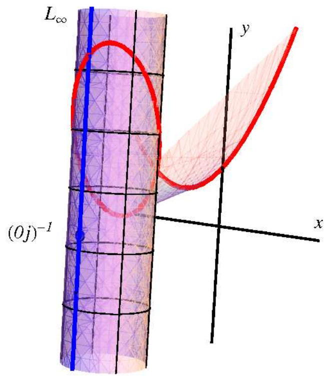

Figure 2. Representation of the dual plane on the Blaschke cylinder.

completely in D to other vertical parabolas and nonvertical lines in D'. Here, D and D' are regions in $\widehat{\mathbb{D}}$ bounded by a single vertical parabola or nonvertical line. By using a preliminary (direct or indirect) Laguerre transformation, we may assume that D is the closed region bounded above by the parabola $y = x^2$; that is, $D = \{y \le x^2\}$.

3.1. ***Preliminary remark.*** Using injectivity, it follows that the number of intersection points of two parabolas contained completely in D is preserved by T. For instance, if parabolas p_0 and p_1 intersect exactly once in D, and are therefore tangent, then the parabolas $T(p_0)$ and $T(p_1)$ also intersect exactly once in D', and are therefore tangent. (If they intersected twice, then one intersection point is the image of distinct points on p_0 and p_1.) Simply put, injectivity means that intersection points of parabolas cannot be created or destroyed.

3.2. ***First normalization.*** By postcomposing T with a direct or indirect Laguerre transformation, we may also assume:

(i) $T(0) = 0$,

(ii) $T\big((0j)^{-1}\big) = (0j)^{-1}$,

(iii) $T : \{y = 0\} \to \{y = 0\}$,

(iv) $T : \{y = x^2\} \to \{y = x^2\}$.

To see this, suppose that the original transformation is T_0, and $T_0(0) = w_0$ and $T_0\big((0j)^{-1}\big) = w_1$. If $\mu_1(w) = (w - w_0)/(w - w_1)$, then $T_1 \overset{\text{def}}{=} \mu_1 \circ T_0$ satisfies $T_1(0) = 0$ and $T_1\big((0j)^{-1}\big) = (0j)^{-1}$. We must determine a further Laguerre transformation μ_2 so that $T \overset{\text{def}}{=} \mu_2 \circ T_1$ satisfies (i)–(iv).

Since $T_1((0j)^{-1}) = (0j)^{-1}$, it already follows that T_1 maps nonvertical lines to nonvertical lines.

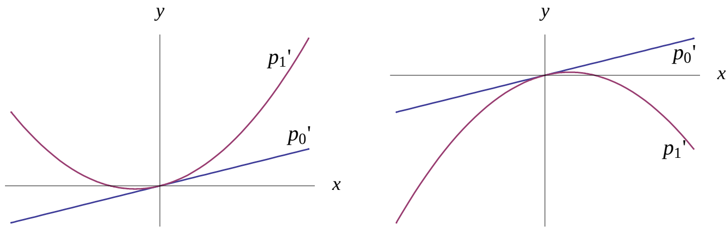

Figure 3. Intermediate configurations during normalization: cases (i) and (ii).

If $p_0 = \{y = 0\}$ and $p_1 = \{y = x^2\}$, then in fact, T_1 maps p_0 to a nonvertical line p_0' through the origin and p_1 to a parabola p_1' tangent to p_0' at the origin. See Figure 3 for the two possible cases. The set of Laguerre transformations that preserve 0 and $(0j)^{-1}$ have the form $\mu_2(z) = az$ or $\mu_2(z) = a\bar{z}$, where $a \in \mathbb{D}$ and Real $(a) \neq 0$.

For case (i), we use a direct transformation $\mu_2(z) = az$ where $\arg(a)$ is chosen so that $\mu_2(p_0') = \{y = 0\}$ and then $|a|$ is chosen so that $\mu_2(p_1') = \{y = x^2\}$. For case (ii), we use an indirect transformation $\mu_2(z) = a\bar{z}$ where the initial conjugation results in a configuration like case (i), and then a is chosen as just described. In both cases, $T \stackrel{\text{def}}{=} \mu_2 \circ T_1$ satisfies (i)–(iv), and we have exhausted our supply of Laguerre transformations.

3.3. ***Parallel lines.*** Recall that the nonvertical lines are exactly those parabolas that intersect L_∞ at the point $(0j)^{-1}$. After the first normalization, T preserves $(0j)^{-1}$, so it follows that T maps nonvertical lines to other nonvertical lines. Since T also preserves 0, it follows that T maps lines through the origin to other lines through the origin. (We used this fact in the second step of the normalization in Section 3.2.) Finally, parallel nonvertical lines intersect exactly once — the intersection occurs at $(0j)^{-1}$. Since T preserves this point, and since T preserves the number of intersection points among parabolas (Section 3.1), it follows that T maps parallel nonvertical lines to other parallel nonvertical lines. As a special case, T preserves $\{y = 0\}$, and so it also follows that T maps horizontal lines to horizontal lines.

3.4. ***Inscribed and circumscribed parabolas for a special unbounded polygon.*** It is a rather curious fact that the parabolas $y = x^2$ and $y = x^2 - 1/4$ arise as inscribed and circumscribed parabolas for a special unbounded polygon. The polygon is constructed from the lines tangent to the parabola $y = x^2$ at points that have integer valued coordinates. See Figure 4. Clearly, the parabola $y = x^2$ inscribes the polygon. To demonstrate that $y = x^2 - 1/4$ circumscribes the polygon, we show that

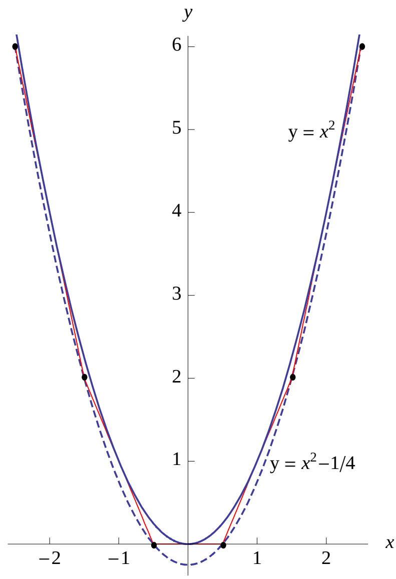

Figure 4. Inscribed and circumscribed parabolas for a special polygon.

the points of intersection of consecutive tangent lines to $y=x^2$ lie on its graph. The line tangent at (k, k^2) has equation $y = 2kx - k^2$. An easy calculation then shows that the lines tangent at (k, k^2) and $(k+1, (k+1)^2)$ intersect at $(k+1/2, k^2+k)$. Another easy calculation shows this point lies on $y = x^2 - 1/4$.

3.5. ***Second normalization.*** By further composing with a nonisotropic dilation $d_\lambda : (x, y) \to (\lambda x, \lambda^2 y)$, we assume $T(1, 1) = (1, 1)$. This is possible for the following reason. If after the first normalization the transformation is T_0, then $T_0(1, 1) = (\rho, \rho^2)$ for $0 \neq \rho \in \mathbb{R}$ since T_0 preserves $\{y = x^2\}$. If $\lambda = \rho^{-1}$ then $T \overset{\text{def}}{=} d_\lambda \circ T_0$ preserves $(1, 1)$. Furthermore, d_λ preserves 0, $(0j)^{-1}$, $\{y = 0\}$, and $\{y = x^2\}$. So T continues to satisfy conditions (i)–(iv) from Section 3.2.

Given that T maps horizontal lines to horizontal lines (Section 3.3) and now $T(1, 1) = (1, 1)$, it follows that T preserves $\{y = 1\}$. But T also preserves $\{y = x^2\}$, so it follows that T preserves both intersection points. In particular, $T(-1, 1) = (-1, 1)$.

Following the two normalizations (which identify the Laguerre transformation and nonisotropic dilation) we mention that the proof of Theorem 2 will be complete once we show that T is the identity transformation. We first prove that T reproduces the configuration in Figure 4.

To do this, notice that $y = x^2$ and $y = x^2 - 1/4$ are tangent parabolas — they intersect only at the point $-(1j)^{-1} \in L_\infty$. Since T preserves $\{y = x^2\}$, and since intersection points cannot be created or destroyed (Section 3.1), it follows that T transforms the parabola $y=x^2-1/4$ to a parabola $y=rx^2+sx+t$ for $r, s, t \in \mathbb{R}$ with $s^2-4(r-1)t=0$. (This is the required condition for a single intersection with $y=x^2$.)

Next, T maps tangent parabolas to tangent parabolas (Section 3.1) and nonvertical line segments to nonvertical line segments (Section 3.3). It also preserves $\{y=x^2\}$ as well as the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$. It follows that the parabola $y=rx^2+sx+t$ must contain the points of intersection of consecutive lines tangent to $y=x^2$ at (x_k, x_k^2) for a sequence of real numbers

$$\dots, x_{-2}, x_{-1}=-1, x_0=0, x_1=1, x_2, \dots.$$

We will verify that together with the required condition for single intersection, this demands $r=1$, $s=0$, $t=-1/4$, and $x_k=k$ for $k \in \mathbb{Z}$.

Notice that the line tangent to $y=x^2$ at (x_k, x_k^2) has equation $y=2x_kx-x_k^2$. From this, one can check that the point of intersection of the lines tangent to $y=x^2$ at (x_k, x_k^2) and (x_{k+1}, x_{k+1}^2) is $\bigl((x_k+x_{k+1})/2, x_kx_{k+1}\bigr)$. Setting $k=-1$ and $k=0$, this means the parabola $y=rx^2+sx+t$ must contain points $(-1/2, 0)$ and $(1/2, 0)$. Together with the condition $s^2-4(r-1)t=0$, this requires $r=1$, $s=0$, and $t=-1/4$. (The other possibility yields the parabola $y=0$. This is ruled out by injectivity.)

At this point it is established that the parabola $y=x^2-1/4$ contains the intersection points $\bigl((x_k+x_{k+1})/2, x_kx_{k+1}\bigr)$. After some algebra, this can be restated as $1=(x_{k+1}-x_k)^2$. So the x_k are evenly spaced with $x_k=k$ for $k=-1, 0, 1$. By injectivity, it follows that $x_k=k$ for $k \in \mathbb{Z}$.

3.6. *Preservation of a dense subset of $y=x^2$.* It follows from Section 3.5 that the normalized transformation T preserves the points

$$(k, k^2) \quad \text{for } k=0, \pm 1, \pm 2, \pm 3, \dots.$$

Here we show as well that T preserves all points on $y=x^2$ whose coordinates are dyadic rational, that is, the coordinates have the form $k \cdot 2^{-q}$ for $k, q \in \mathbb{Z}$.

To do this, we construct another polygon like the one in Section 3.4. In particular, for fixed $q \geq 1$, we draw lines tangent to $y=x^2$ at the points $(k\,2^{-q}, k^2\,2^{-2q})$ for $k=0, \pm 1, \pm 2, \pm 3, \dots$. It is a simple matter to check that the points of intersection of the consecutive tangent lines are $\bigl((2k+1)2^{-q-1}, k(k+1)2^{-2q}\bigr)$, and the parabola $y=x^2-2^{-2q-2}$ contains these intersection points. As in Section 3.5, T must transform this configuration to one consisting of the parabola $y=x^2$, lines tangent to $y=x^2$ at points (x_k, x_k^2) for a sequence of real numbers

$$\dots, x_{-2}, x_{-1}, x_0=0, x_1, x_2, \dots,$$

and a parabola $y = rx^2 + sx + t$ that has a single intersection with $y = x^2$. The parabola $y = rx^2 + sx + t$ must also contain the intersection points of the consecutive lines tangent to $y = x^2$ at the (x_k, x_k^2). Since T was normalized so that $T(1, 1) = (1, 1)$ and $T(-1, 1) = (-1, 1)$, we know that $x_{-2^q} = -1$ and $x_{2^q} = 1$. The claim will be proved if we show that $r = 1$, $s = 0$, $t = -2^{-2q-2}$, and $x_k = k\, 2^{-q}$ for $k \in \mathbb{Z}$.

As things are arranged, $T(k\, 2^{-q}, k^2\, 2^{-2q}) = (x_k, x_k^2)$ for $k \in \mathbb{Z}$. Since T maps horizontal lines to horizontal lines, it follows that $x_k^2 = x_{-k}^2$, and in particular, $x_{-k} = -x_k$. This means that the target configuration must be symmetric with respect to the y-axis. Therefore, $s = 0$. Already, the single intersection of $y = x^2$ with $y = rx^2 + sx + t$ requires $s^2 - 4(r - 1)t = 0$, so now $r = 1$ or $t = 0$. The case $t = 0$ is ruled out else T transforms the parabola $y = x^2 - 2^{-2q-2}$ to a parabola $y = rx^2$ that intersects $y = 0$ exactly once (if $r \neq 0$) or else infinitely many times (if $r = 0$).

We conclude that T transforms $y = x^2 - 2^{-2q-2}$ to a parabola $y = x^2 + t$ for some $0 \neq t \in \mathbb{R}$. In fact, since T preserves $\{y = 0\}$, it must be that $t < 0$. (This also uses Section 3.1.)

As in Section 3.5, the intersection points of the lines tangent to $y = x^2$ at (x_k, x_k^2) have the form $\big((x_k + x_{k+1})/2, x_k x_{k+1}\big)$ and they lie on $y = x^2 + t$. It follows that $x_k x_{k+1} = (x_k + x_{k+1})^2/4 + t$, or equivalently, $t = -(x_{k+1} - x_k)^2/4$ for $k \in \mathbb{Z}$. In particular, the x_k are evenly spaced. Since $x_0 = 0$ and $x_{2^q} = 1$, it follows that the distance from x_k to x_{k+1} is 2^{-q}, and therefore, $x_k = k\, 2^{-q}$ for $k \geq 1$. (This also uses injectivity.) The same kind of argument applies for $k \leq -1$. Finally, one finds easily that $t = -(x_{k+1} - x_k)^2/4 = -2^{-2q-2}$.

3.7. ***Preservation of a dense subset of $y < x^2$.*** Guided by Carathéodory's argument [Carathéodory 1937, page 576], we now show that T preserves a set of points that is dense in $y < x^2$. To do this, we take all lines that are tangent to the parabola $y = x^2$ at points whose coordinates are dyadic rational. By the previous subsection, T preserves these points of tangency (Section 3.6) along with the parabola $y = x^2$ (Section 3.2), so it also preserves the lines tangent to $y = x^2$ at these points (Section 3.1, Section 3.3). It then follows that T preserves each point of intersection of these tangent lines. These intersection points form a dense subset of $y < x^2$.

The set includes, in particular, the points $(-.5, -2)$, $(0, -1)$, and $(+.5, -2)$. (They arise as the intersection points of the lines tangent at $x = -2, -1, +1, +2$.) Since T preserves these points, and since T maps vertical parabolas and nonvertical lines to vertical parabolas and nonvertical lines, it follows that T preserves the unique vertical parabola containing these points. In particular, T preserves the parabola $y = -4x^2 - 1$.

3.8. ***Completion of the proof of Theorem 2.*** We next show that T preserves each point of the parabola $y = x^2$.

Consider the alternative. Since T preserves $\{y = x^2\}$, the alternative is that there is a (nondyadic) real number b and $\tau \notin \{0, 1\}$ so that $T(b, b^2) = (\tau b, \tau^2 b^2)$. By once more replaying the arguments from Section 3.5–3.6, it would follow that $T(d \cdot b, d^2 \cdot b^2) = (d \cdot \tau b, d^2 \cdot \tau^2 b^2)$ for dyadic rational d. Moreover, T would map the line tangent to $y = x^2$ at $x = d \cdot b$ to the line tangent to $y = x^2$ at $x = d \cdot \tau b$.

Following the argument of Section 3.7, this determines how T would act on the set of points that arise as the intersection points of lines tangent to $y = x^2$ at $x = d \cdot b$ for d dyadic rational. On this dense set of points, $T(x, y) = (\tau x, \tau^2 y)$.

Consider now that d is a fixed (but arbitrary) dyadic rational number. The lines tangent to $y = x^2$ at $x = d \cdot b$ and $x = -2d \cdot b$ intersect at $(-d \cdot b/2, -2d^2 \cdot b^2)$, and the lines tangent to $y = x^2$ at $x = -d \cdot b$ and $x = 2d \cdot b$ intersect at $(+d \cdot b/2, -2d^2 \cdot b^2)$. These points determine the horizontal line $y = -2d^2 \cdot b^2$. As T maps horizontal lines to horizontal lines (Section 3.3), and since the intersection points belong to the set on which $T(x, y) = (\tau x, \tau^2 y)$, it follows that T would map the horizontal line $y = -2d^2 \cdot b^2$ to the horizontal line $y = -2d^2 \cdot \tau^2 b^2$.

Here lies the contradiction. If $\tau^2 < 1$, choose a dyadic d so that

$$(2b^2)^{-1} < d^2 < (2b^2\tau^2)^{-1}.$$

Then the parabola $y = -4x^2 - 1$ intersects the line $y = -2d^2 \cdot b^2$ twice, but after the action of T, the parabola $y = -4x^2 - 1$ does not intersect the line $y = -2d^2 \cdot \tau^2 b^2$. This violates Section 3.1.

Similarly, if $\tau^2 > 1$, choose a dyadic d so that $(2b^2\tau^2)^{-1} < d^2 < (2b^2)^{-1}$. Then the parabola $y = -4x^2 - 1$ does not intersect the line $y = -2d^2 \cdot b^2$, but after the action of T, the parabola $y = -4x^2 - 1$ intersects the line $y = -2d^2 \cdot \tau^2 b^2$. Again this violates Section 3.1.

There is the remaining case $\tau = -1$. For this we identify an asymmetric parabola preserved by T. For instance, the lines tangent to $y = x^2$ at $x = 0, 0.5, 2, 2.5, 4$ determine intersection points $(1, 0)$, $(2, 0)$, and $(1.5, 1.25)$. These points determine the parabola $y = -5x^2 + 15x - 10$ that is preserved by T. Next, we choose a dyadic d so that $|4bd - 5| < 5/\sqrt{3}$. This condition guarantees that the line tangent to $y = x^2$ at $x = d \cdot b$ does not intersect $y = -5x^2 + 15x - 10$, but the line tangent at $x = -d \cdot b$ does intersect $y = -5x^2 + 15x - 10$. Under the action of T, the tangent line at $x = d \cdot b$ would be mapped to the tangent line at $x = -d \cdot b$, creating an intersection with $y = -5x^2 + 15x - 10$. Again the contradiction.

We conclude that T preserves each point of $y = x^2$, and it follows from the argument in Section 3.7 that T preserves all points below $y = x^2$, since each such point can be expressed as the intersection of lines tangent to $y = x^2$. That is, T acts identically on $y \leq x^2$, and following the remark in Section 3.5, this completes the proof of Theorem 2.

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