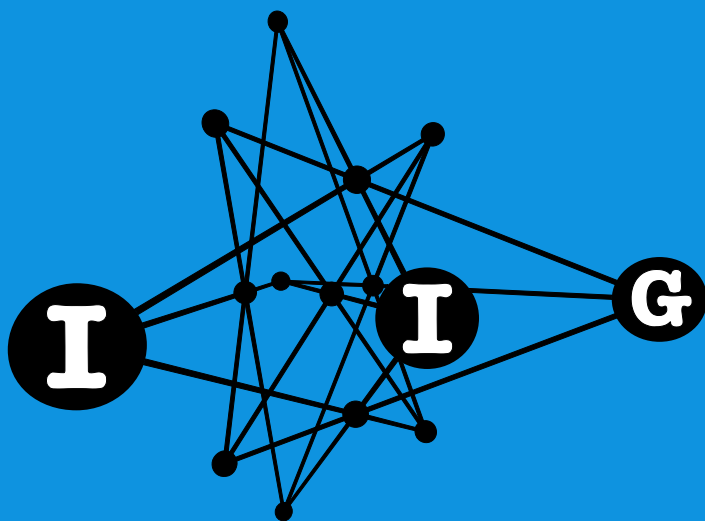


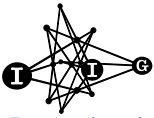
Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial



Automorphisms of (affine) $SL(2, q)$ -unitals

Verena Möhler



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$SL(2, q)$ -unitals are unitals of order q admitting a regular action of $SL(2, q)$ on the complement of some block. They can be obtained from affine $SL(2, q)$ -unitals via parallelisms. We compute a sharp upper bound for automorphism groups of affine $SL(2, q)$ -unitals and show that exactly two parallelisms are fixed by all automorphisms. In nonclassical $SL(2, q)$ -unitals obtained as closures of affine $SL(2, q)$ -unitals via those two parallelisms, we show that there is one block fixed under the full automorphism group.

1. Preliminaries

A *unital of order n* is a 2 - $(n^3+1, n+1, 1)$ design, i. e. an incidence structure with $n^3 + 1$ points, $n + 1$ points on each block and unique joining blocks for any two points. We also consider *affine unitals*, which arise from unitals by removing one block (and all the points on it) and can be completed to unitals via a parallelism on the short blocks. We give an axiomatic description:

Definition 1.1. Let $n \in \mathbb{N}$, $n \geq 2$. An incidence structure $\mathbb{U} = (\mathcal{P}, \mathcal{B}, I)$ is called an *affine unital of order n* if:

- (AU1) There are $n^3 - n$ points.
- (AU2) Each block is incident with either n or $n + 1$ points. The blocks incident with n points will be called *short blocks* and the blocks incident with $n + 1$ points will be called *long blocks*.
- (AU3) Each point is incident with n^2 blocks.
- (AU4) For any two points there is exactly one block incident with both of them.

Most of the results in this paper have been obtained in the author's Ph. D. thesis [9], where detailed arguments can be found for some statements left to the reader here.

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(AU5) There exists a *parallelism* on the short blocks, meaning a partition of the set of all short blocks into $n + 1$ parallel classes of size $n^2 - 1$ such that the blocks of each parallel class are pairwise nonintersecting.

The existence of a parallelism as in (AU5) must explicitly be required (see [9, Example 3.10]). An affine unital \mathbb{U} of order n with parallelism π can be completed to a unital \mathbb{U}^π of order n as follows: For each parallel class, add a new point that is incident with each short block of that class. Then add a single new block $[\infty]^\pi$, incident with the $n + 1$ new points (see [9, Proposition 3.9]). We call \mathbb{U}^π the π -closure of \mathbb{U} .

Although we must require the existence of a parallelism in the definition of an affine unital, this parallelism need not be unique. It is therefore not convenient to require that isomorphisms of affine unitals respect certain parallelisms and we will only ask them to be isomorphisms of the incidence structures. We call two parallelisms π and π' of an affine Unital \mathbb{U} *equivalent* if there is an automorphism of \mathbb{U} which maps π to π' .

Given an affine unital \mathbb{U} with parallelisms π and π' , the closures \mathbb{U}^π and $\mathbb{U}^{\pi'}$ are isomorphic with $[\infty]^\pi \mapsto [\infty]^{\pi'}$ exactly if π and π' are equivalent (see [9, Proposition 3.12]). Indeed, two closures \mathbb{U}^π and $\mathbb{U}^{\pi'}$ being isomorphic does not imply the parallelisms π and π' being equivalent in \mathbb{U} :

Example 1.2. Let \mathbb{U} be the affine unital of order 3 (24 points, 30 long blocks and 32 short blocks) indicated in Figure 1.

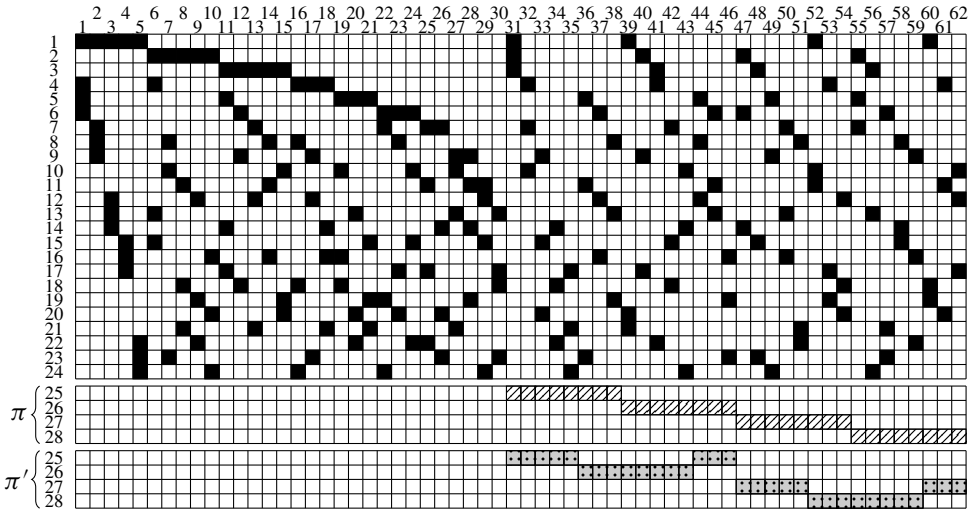


Figure 1. An affine unital with two nonequivalent parallelisms, where the closures are isomorphic.

We consider two parallelisms π and π' of \mathbb{U} , in the figure indicated via the incidences of the four additional points 25, \dots , 28. Those two parallelisms are not equivalent in \mathbb{U} — in fact, $\mathrm{Aut}(\mathbb{U})$ is trivial — but the corresponding closures \mathbb{U}^π and $\mathbb{U}^{\pi'}$ are isomorphic via the isomorphism

$$(1, 16, 23, 10)(2, 11, 15, 19)(4, 9, 13, 14)(5, 22, 18, 24)(6, 27, 20, 25, 8, 26, 17, 28)(12, 21).$$

2. (Affine) $\mathrm{SL}(2, q)$ -Unitals

From now on let p be a prime and $q := p^e$ a p -power. We are interested in a special kind of affine unitals, namely affine $\mathrm{SL}(2, q)$ -unitals. The construction of those affine unitals is due to Grundhöfer, Stroppel and Van Maldeghem [3]. They consider translations of unitals, i. e. automorphisms fixing each block through a given point (the so-called center). Of special interest are unitals of order q where two points are centers of translation groups of order q . In the classical (Hermitian) unital of order q , any two such translation groups generate a group isomorphic to $\mathrm{SL}(2, q)$; see [4, Main Theorem] for further possibilities. The construction of (affine) $\mathrm{SL}(2, q)$ -unitals is motivated by this action of $\mathrm{SL}(2, q)$ on the classical unital.

Let $S \leq \mathrm{SL}(2, q)$ be a subgroup of order $q + 1$ and let $T \leq \mathrm{SL}(2, q)$ be a Sylow p -subgroup. Recall that T has order q (and thus trivial intersection with S), that any two conjugates $T^h := h^{-1}Th$, $h \in \mathrm{SL}(2, q)$, have trivial intersection unless they coincide and that there are $q + 1$ conjugates of T .

Consider a collection \mathcal{D} of subsets of $\mathrm{SL}(2, q)$ such that each set $D \in \mathcal{D}$ contains $\mathbb{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, that $\#D = q + 1$ for each $D \in \mathcal{D}$, and the following properties hold:

(Q) For each $D \in \mathcal{D}$, the map

$$(D \times D) \setminus \{(x, x) \mid x \in D\} \rightarrow \mathrm{SL}(2, q), \quad (x, y) \mapsto xy^{-1},$$

is injective, i. e. the set $D^* := \{xy^{-1} \mid x, y \in D, x \neq y\}$ contains $q(q + 1)$ elements.

(P) The system consisting of $S \setminus \{\mathbb{1}\}$, all conjugates of $T \setminus \{\mathbb{1}\}$ and all sets D^* with $D \in \mathcal{D}$ forms a partition of $\mathrm{SL}(2, q) \setminus \{\mathbb{1}\}$.

Set

$$\mathcal{P} := \mathrm{SL}(2, q),$$

$$\mathcal{B} := \{Sg \mid g \in \mathrm{SL}(2, q)\} \cup \{T^h g \mid h, g \in \mathrm{SL}(2, q)\} \cup \{Dg \mid D \in \mathcal{D}, g \in \mathrm{SL}(2, q)\}$$

and let the incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$ be containment.

We call the incidence structure $\mathbb{U}_{S, \mathcal{D}} := (\mathcal{P}, \mathcal{B}, I)$ an *affine* $\mathrm{SL}(2, q)$ -*unital*. Each affine $\mathrm{SL}(2, q)$ -unital is indeed an affine unital of order q , see [9, Prop. 3.15]. If π is a parallelism on the short blocks of an affine $\mathrm{SL}(2, q)$ -unital $\mathbb{U}_{S, \mathcal{D}}$, we call the π -closure $\mathbb{U}_{S, \mathcal{D}}^\pi$ an $\mathrm{SL}(2, q)$ -(π -)unital.

In any affine $\mathrm{SL}(2, q)$ -unital, the set of short blocks is the set of all right cosets of the Sylow p -subgroups, i. e.

$$\{Tg \mid T \in \mathfrak{P}, g \in \mathrm{SL}(2, q)\},$$

where \mathfrak{P} denotes the set of the $q + 1$ Sylow p -subgroups of $\mathrm{SL}(2, q)$. Note that each right coset Tg is a left coset gT^s of a conjugate of T . A parallelism as in (AU5) means a partition of the set of short blocks into $q + 1$ sets of $q^2 - 1$ pairwise nonintersecting cosets. For each prime power q , there are hence two obvious parallelisms, namely partitioning the set of short blocks into the sets of *right* cosets or into the sets of *left* cosets of the Sylow p -subgroups. We name those two parallelisms “flat” and “natural”, respectively, and denote them by the corresponding musical signs:

$$\flat := \{\{Tg \mid g \in \mathrm{SL}(2, q)\} \mid T \in \mathfrak{P}\}, \quad \natural := \{\{gT \mid g \in \mathrm{SL}(2, q)\} \mid T \in \mathfrak{P}\}.$$

Example 2.1.

- (a) For each prime power q we may choose $S = C$ to be cyclic and \mathcal{H} a set of blocks through $\mathbb{1}$ such that $\mathbb{U}_{C, \mathcal{H}}$ is isomorphic to the affine part of the classical unital and the closure $\mathbb{U}_{C, \mathcal{H}}^\natural$ is isomorphic to the classical unital. We call $\mathbb{U}_{C, \mathcal{H}}$ the *classical affine* $\mathrm{SL}(2, q)$ -unital and $\mathbb{U}_{C, \mathcal{H}}^\natural$ the *classical* $\mathrm{SL}(2, q)$ -unital. See [3, Example 3.1] or [9, Section 3.2.2] for details.

In [3, Example 5.5], Grundhöfer, Stroppel and Van Maldeghem state that the unital $\mathbb{U}_{C, \mathcal{H}}^\flat$ coincides with a unital described by Grüning, that can be embedded in a Hall plane and in its dual, see [5].

- (b) Several nonclassical affine $\mathrm{SL}(2, q)$ -unitals are known, namely one of order 4, described in [3], and three of order 8, described in [10, Section 3].
- (c) There are also other parallelisms than \flat and \natural , see [11, Sections 3.1–3.3]. The construction of $\mathrm{SL}(2, q)$ -unitals inspired Nagy and Mezőfi to a method of constructing new incidence structures from old ones by removing a block and attaching it again in a different way, see [8]. By computer search, they found plenty parallelisms for affine unitals of orders 3 and 4, respectively. Among the parallelisms of affine $\mathrm{SL}(2, q)$ -unitals, we show in [Theorem 3.7](#) that \flat and \natural are the only ones being preserved by all possible automorphisms of affine $\mathrm{SL}(2, q)$ -unitals.

3. Automorphisms

Concerning automorphisms of (affine) $SL(2, q)$ -unitals, we note first that on any affine $SL(2, q)$ -unital $\cup_{S, \mathcal{D}}$, the group $SL(2, q)$ acts as group of automorphisms by multiplication from the right. Each automorphism of $SL(2, q)$ induces a permutation on the point set of $\cup_{S, \mathcal{D}}$, while it need not leave the block set invariant. Since any automorphism of affine unitals maps short blocks to short blocks, we will first consider the incidence structure \mathfrak{S} given by the points and short blocks of any affine $SL(2, q)$ -unital. Recall that the short blocks are the same in every affine $SL(2, q)$ -unital, independent of the choice of S and the set \mathcal{D} .

3.1. Automorphisms of the geometry of short blocks. We use an embedding of \mathfrak{S} into the classical generalized quadrangle $Q(4, q)$. This quadrangle is the polar space given by the equation

$$x_1x_3 + x_2x_4 + x_5^2 = 0$$

in homogeneous coordinates for the projective space $PG(4, q)$. The intersection of $Q(4, q)$ with the hyperplane $x_5 = 0$ of $PG(4, q)$ is a geometric hyperplane H of the generalized quadrangle, isomorphic to the classical generalized quadrangle $Q(3, q)$, which is the polar space given by the equation

$$x_1x_3 + x_2x_4 = 0$$

in homogeneous coordinates for the projective space $PG(3, q)$. The geometry of short blocks \mathfrak{S} is isomorphic to the complement $Q(4, q) \setminus H$, which follows basically from [13, 10.7.8] and [15, Theorem 1.1] (see [9, Section 4.1] for details). In any thick classical generalized quadrangle, the automorphisms of a hyperplane complement are exactly the automorphisms of the quadrangle stabilizing the hyperplane (see [12, Lemma 2.3]). Thus,

$$\text{Aut}(\mathfrak{S}) \cong \text{Aut}(Q(4, q))_H.$$

Compute $\text{Aut}(Q(4, q))_H \cong O(f) \times \text{Aut}(\mathbb{F}_q)$, where f is the quadratic form

$$f: \mathbb{F}_q^4 \rightarrow \mathbb{F}_q, x = (x_1, \dots, x_4) \mapsto x_1x_3 + x_2x_4,$$

and $O(f) = \{A \in GL(4, q) \mid f(xA) = f(x) \forall x \in \mathbb{F}_q^4\}$ is the corresponding orthogonal group (again, see [9, Section 4.1] for details).

We thus know the isomorphism type (and, in particular, the order) of $\text{Aut}(\mathfrak{S})$ and may now study the action of $\text{Aut}(\mathfrak{S})$ on the incidence structure \mathfrak{S} . We already know that $SL(2, q)$ acts via right multiplication as a group of automorphisms on \mathfrak{S} . Applying automorphisms of $SL(2, q)$ to the point set of \mathfrak{S} also induces automorphisms of \mathfrak{S} and so does inversion, since the set of short blocks of any affine $SL(2, q)$ -unital is given by the set of *all* right cosets of the Sylow p -subgroups

of $\mathrm{SL}(2, q)$, and every left coset of a Sylow p -subgroup T is a right coset of a conjugate of T . (Recall $(Tg)^{-1} = g^{-1}T = T^g g^{-1}$.)

Let $R := \{\rho_h \mid h \in \mathrm{SL}(2, q)\} \leq \mathrm{Aut}(\mathfrak{S})$, where $\rho_h \in R$ acts on \mathfrak{S} by right multiplication with $h \in \mathrm{SL}(2, q)$. Let further $\mathfrak{A} \leq \mathrm{Aut}(\mathfrak{S})$ denote the subgroup given by automorphisms of $\mathrm{SL}(2, q)$ and $I \leq \mathrm{Aut}(\mathfrak{S})$ the cyclic subgroup of order 2 given by inversion. For each $a \in \mathrm{GL}(2, q)$, conjugation by a induces an automorphism $\gamma_a \in \mathfrak{A}$.

Lemma 3.1. $\mathrm{Aut}(\mathfrak{S}) = (\mathfrak{A} \times I) \cdot R$.

Proof. From the above, the product $\mathfrak{A} \cdot I \cdot R$ is a subset of $\mathrm{Aut}(\mathfrak{S})$. The product $\mathfrak{A} \cdot I$ is direct (and in particular a subgroup of $\mathrm{Aut}(\mathfrak{S})$), since \mathfrak{A} and I commute and have trivial intersection. The group R has trivial intersection with $\mathfrak{A} \times I$, since \mathfrak{A} and I fix the point $\mathbb{1}$, while R acts regularly on the point set. Hence we may compute

$$\begin{aligned} \#((\mathfrak{A} \times I) \cdot R) &= \#\mathfrak{A} \cdot \#I \cdot \#R \\ &= \#\mathrm{PGL}(2, q) \cdot 2 \cdot \#\mathrm{SL}(2, q) \\ &= e(q-1)q(q+1) \cdot 2 \cdot (q-1)q(q+1) \\ &= 2e(q-1)^2q^2(q+1)^2. \end{aligned}$$

We have already noted that $\mathrm{Aut}(\mathfrak{S})$ is isomorphic to $\mathrm{O}(f) \times \mathrm{Aut}(\mathbb{F}_q)$. But $\#\mathrm{O}(f) = 2(q-1)^2q^2(q+1)^2$, see [14, p. 141] or [16, Theorems 6.21 and 7.23 with $v = 2$ and $\delta = 0$], and the lemma follows. \square

Lemma 3.2. \mathfrak{A} normalizes R in the full automorphism group of \mathfrak{S} .

Proof. Let $\alpha \in \mathfrak{A}$, $\rho_h \in R$ and let $x \in \mathrm{SL}(2, q)$ be a point of \mathfrak{S} . Then

$$x \cdot (\alpha^{-1} \rho_h \alpha) = ((x \cdot \alpha^{-1})h) \cdot \alpha = x(h \cdot \alpha) = x \cdot \rho_{h \cdot \alpha}. \quad \square$$

Since \mathfrak{A} and R have trivial intersection, we thus know that the product of \mathfrak{A} and R is semidirect.

3.2. Automorphisms of affine $\mathrm{SL}(2, q)$ -unitals. We use our knowledge of the full automorphism group of \mathfrak{S} to compute automorphism groups of (affine) $\mathrm{SL}(2, q)$ -unitals.

Theorem 3.3. Let $q \geq 3$ and let $\mathbb{U}_{S, \mathcal{D}}$ and $\mathbb{U}_{S', \mathcal{D}'}$ be affine $\mathrm{SL}(2, q)$ -unitals.

- (a) Let $\psi: \mathbb{U}_{S, \mathcal{D}} \rightarrow \mathbb{U}_{S', \mathcal{D}'}$ be an isomorphism. Then $\psi = \alpha \rho_h$ with $\rho_h \in R$ and $\alpha \in \mathfrak{A}$ such that $S \cdot \alpha = S'$.
- (b) $\mathrm{Aut}(\mathbb{U}_{S, \mathcal{D}}) \leq \mathfrak{A}_S \times R$. \square

Proof. (a) Each isomorphism of affine $\mathrm{SL}(2, q)$ -unitals maps short blocks to short blocks and is hence an automorphism of \mathfrak{S} , since the incidence structure of the short blocks is the same in every affine $\mathrm{SL}(2, q)$ -unital. We thus know $\psi \in \mathrm{Aut}(\mathfrak{S}) = (\mathfrak{A} \times I) \cdot R$. Let $\psi = \alpha \iota \rho_h$ with $\iota \in I$ inversion or identity. Then

$$S \cdot \psi = S \cdot (\alpha \iota \rho_h) = (S \cdot \alpha)h,$$

since S is a group and α an automorphism of $\mathrm{SL}(2, q)$. Since right multiplication by $\mathrm{SL}(2, q)$ induces automorphisms in every affine $\mathrm{SL}(2, q)$ -unital, $S \cdot \alpha$ is a block through $\mathbb{1}$ in $\mathbb{U}_{S', \mathcal{D}'}$ and is a subgroup of $\mathrm{SL}(2, q)$ of order $q + 1$. The only block with these properties is S' and hence $S \cdot \alpha = S'$. Assume ι to be inversion and choose $g \in \mathrm{SL}(2, q)$ with $g \cdot \alpha \notin N(S')$. Note that such an element g exists, since $q \geq 3$. Then

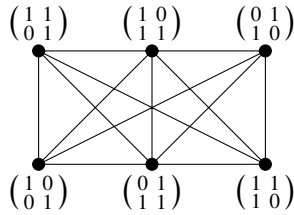
$$(Sg) \cdot \psi = (Sg) \cdot (\alpha \iota \rho_h) = ((S \cdot \alpha)(g \cdot \alpha))^{-1}h = (S')^{g \cdot \alpha}(g \cdot \alpha)^{-1}h.$$

Sg is a block of $\mathbb{U}_{S, \mathcal{D}}$ and hence $(S')^{g \cdot \alpha}$ is a block of $\mathbb{U}_{S', \mathcal{D}'}$ with $\mathbb{1}$ in $(S')^{g \cdot \alpha}$ and $(S')^{g \cdot \alpha} \leq \mathrm{SL}(2, q)$ is a subgroup of order $q + 1$. But $(S')^{g \cdot \alpha} \neq S'$ by the choice of g , a contradiction.

(b) Directly from [Lemma 3.2](#) and (a). □

Remark 3.4. The classical affine $\mathrm{SL}(2, q)$ -unital $\mathbb{U}_{C, \mathcal{H}}$ admits the whole group $\mathfrak{A}_C \times R$ as automorphism group (see [[9](#), Proposition 4.6]). Hence, $\mathfrak{A}_S \times R$ is a sharp upper bound for the automorphism group of any affine $\mathrm{SL}(2, q)$ -unital $\mathbb{U}_{S, \mathcal{D}}$ of order $q \geq 3$.

Example 3.5. It is easily seen that there is only one isomorphism type of affine unitals of order 2. Labeling the points with elements of $\mathrm{SL}(2, 2)$, we get the classical affine $\mathrm{SL}(2, 2)$ -unital $\mathbb{U}_{C, \mathcal{H}}$ with $C = \left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix} \right) \right)$ and \mathcal{H} the empty set:



Obviously, inversion induces an automorphism of this affine unital (interchanging the points $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$) and we get $\mathrm{Aut}(\mathbb{U}_{C, \mathcal{H}}) = (\mathfrak{A}_C \times I) \cdot R \cong \mathfrak{A}_C \times R$ for $q = 2$.

3.3. Automorphisms of $\mathrm{SL}(2, q)$ -unitals with parallelism \mathfrak{b} or \mathfrak{h} . We now take a closer look at $\mathrm{SL}(2, q)$ -unitals with parallelism \mathfrak{b} or \mathfrak{h} . Recall the definitions

$$\mathfrak{b} := \{ \{Tg \mid g \in \mathrm{SL}(2, q)\} \mid T \in \mathfrak{P} \} \quad \text{and} \quad \mathfrak{h} := \{ \{gT \mid g \in \mathrm{SL}(2, q)\} \mid T \in \mathfrak{P} \},$$

where \mathfrak{P} denotes the set of all Sylow p -subgroups of $\mathrm{SL}(2, q)$. Considering the action of R (i. e. right multiplication by $\mathrm{SL}(2, q)$), we see that both \flat and \natural are invariant under this action. On \flat , right multiplication by $\mathrm{SL}(2, q)$ fixes each parallel class, while on \natural right multiplication with $h \in \mathrm{SL}(2, q)$ maps any left coset gT to the left coset ghT^h and hence the group R acts on the parallel classes of \natural via conjugation on the Sylow p -subgroups. Since the group \mathfrak{A} acts via automorphisms of $\mathrm{SL}(2, q)$, both parallelisms \flat and \natural are fixed under the action of \mathfrak{A} . Note that inversion interchanges \flat with \natural .

Since the stabilizer of the block $[\infty] = [\infty]^\pi$ in the full automorphism group of the π -closure of any affine unital \mathbb{U} equals the group of automorphisms of \mathbb{U} fixing the parallelism π , we get:

Corollary 3.6. Let $q \geq 3$, $\pi \in \{\flat, \natural\}$ and let $\mathbb{U}_{S, \mathcal{D}}$ be an affine $\mathrm{SL}(2, q)$ -unital. Then

$$\mathrm{Aut}(\mathbb{U}_{S, \mathcal{D}}^\pi)_{[\infty]} = \mathrm{Aut}(\mathbb{U}_{S, \mathcal{D}})$$

and in particular $\mathrm{Aut}(\mathbb{U}_{C, \mathcal{H}}^\pi)_{[\infty]} = \mathfrak{A}_C \times R$ for $q \geq 3$. \square

The two parallelisms \flat and \natural are indeed the only parallelisms in any affine $\mathrm{SL}(2, q)$ -unital that are preserved under the action of R :

Theorem 3.7. Let $\mathbb{U}_{S, \mathcal{D}}$ be an affine $\mathrm{SL}(2, q)$ -unital with parallelism π such that right multiplication by $\mathrm{SL}(2, q)$ preserves π . Then $\pi \in \{\flat, \natural\}$.

Proof. Let $T \in \mathfrak{P}$ and let $[T]$ denote the parallel class of π containing the short block T . Since right multiplication with $t \in T$ fixes the block T , it fixes also the parallel class $[T]$. We distinguish two cases.

Assume first that $[T]$ contains exactly the $q^2 - 1$ left cosets of T . Then for each $h \in \mathrm{SL}(2, q)$, we have

$$[T] \cdot h = \{gT \mid g \in \mathrm{SL}(2, q)\} \cdot h = \{gTh \mid g \in \mathrm{SL}(2, q)\} = \{ghT^h \mid g \in \mathrm{SL}(2, q)\},$$

which equals the set of left cosets of T^h and must be a parallel class of π . Hence, each parallel class of π is the complete set of left cosets of some Sylow p -subgroup, meaning $\pi = \natural$.

Assume now that $[T]$ contains at least one short block $T^h g = gT^{hg}$ which is not a left coset of T , i. e. $T^{hg} \neq T$. Then right multiplication with T as well as right multiplication with T^{hg} fixes a block in $[T]$ and hence also the parallel class $[T]$. Since any two different Sylow p -subgroups generate $\mathrm{SL}(2, q)$, we have $\langle T \cup T^{hg} \rangle = \mathrm{SL}(2, q)$. Hence, right multiplication with $\mathrm{SL}(2, q)$ stabilizes $[T]$ and $[T]$ equals the set of right cosets of T . The same reasoning works for each parallel class and thus $\pi = \flat$. \square

Definition 3.8. A translation with center c of a unital \mathbb{U} is an automorphism of \mathbb{U} that fixes the point c and each block through c . The group of all translations with center c will be denoted by $G_{[c]}$.

We call c a *translation center* if $G_{[c]}$ acts transitively on the set of points different from c on any block through c .

Remark 3.9. Let \mathbb{U} be a unital of order q and c a point of \mathbb{U} . Then the group $G_{[c]}$ of all translations of \mathbb{U} with center c acts semiregularly on the points of \mathbb{U} different from c (see [2, Theorem 1.3]). Hence, $\#G_{[c]} \leq q$ and c is a translation center exactly if $\#G_{[c]} = q$.

In any $SL(2, q)$ - π -unital $\mathbb{U}_{S, \mathcal{D}}^\pi$, we label the points on the block $[\infty]$ with the Sylow p -subgroups in such a way that each (affine short) block $T \in \mathfrak{P}$ through $\mathbb{1}$ is incident with the point $T \in [\infty]$.

Lemma 3.10. Let $\mathbb{U}_{S, \mathcal{D}}^\natural$ be an $SL(2, q)$ - \natural -unital. For each $T \in \mathfrak{P}$, the point $T \in [\infty]$ is a translation center with $G_{[T]} = R_T := \{\rho_t \mid t \in T\}$.

Proof. For each $T \in \mathfrak{P}$, the set of blocks through the point T in $\mathbb{U}_{S, \mathcal{D}}^\natural$ equals $\{[\infty]\} \cup \{gT \mid g \in SL(2, q)\}$. Hence, R_T obviously is a group of translations of $\mathbb{U}_{S, \mathcal{D}}^\natural$ with center T . Since $\#R_T = q$, the statement follows (recall Remark 3.9). \square

We use this statement on translations and a theorem of Grundhöfer, Stroppel and Van Maldeghem [2] to show that the block $[\infty]$ is fixed by every automorphism in any nonclassical $SL(2, q)$ - \natural -unital.

Proposition 3.11. Let $\mathbb{U}_{S, \mathcal{D}}^\natural$ be a nonclassical $SL(2, q)$ - \natural -unital. Then

$$\text{Aut}(\mathbb{U}_{S, \mathcal{D}}^\natural) = \text{Aut}(\mathbb{U}_{S, \mathcal{D}}^\natural)_{[\infty]}.$$

Proof. Assume that there is an automorphism of $\mathbb{U}_{S, \mathcal{D}}^\natural$ not fixing $[\infty]$. Since all the points on $[\infty]$ are translation centers (see Lemma 3.10), there are thus three noncollinear translation centers of $\mathbb{U}_{S, \mathcal{D}}^\natural$. Then $\mathbb{U}_{S, \mathcal{D}}^\natural$ is the classical unital, as is shown in [2]. \square

Grüning showed that in each Grüning unital $\mathbb{U}_{C, \mathcal{H}}^b$, the block $[\infty]$ is fixed by every automorphism (see [5, Lemma 5.5]). We will extend this statement to every $SL(2, q)$ - b -unital $\mathbb{U}_{S, \mathcal{D}}^b$ of order $q \geq 3$, independent of the group S and the set \mathcal{D} . For order 2, there is only one isomorphism type of unitals, represented by the classical unital. Hence, any unital of order 2 admits a 2-transitive automorphism group and no block is fixed by the full automorphism group.

Our proof of Theorem 3.16 below uses further knowledge about the group $S \leq SL(2, q)$ of order $q + 1$ and its stabilizer in $\text{Aut}(SL(2, q))$. The following statements are verified by using known results about $\text{PSL}(2, q)$, see [9, Section 2.2] for details. Note that if q is odd, then S contains the unique involution $-\mathbb{1} \in SL(2, q)$.

Proposition 3.12 ([9], Proposition 2.5). Let $S \leq SL(2, q)$ be a subgroup of order $q + 1$. Then:

- (a) For $q \not\equiv 3 \pmod{4}$, the group S is cyclic.
- (b) For $q \equiv 3 \pmod{4}$, there are the following possibilities:
 - (i) S is cyclic.
 - (ii) S is a generalized quaternion group and the quotient $S/\{\pm 1\}$ in $\mathrm{PSL}(2, q)$ is a dihedral group of order $\frac{1}{2}(q+1)$.
 - (iii) The quotient $S/\{\pm 1\}$ in $\mathrm{PSL}(2, q)$ is either isomorphic to A_4 with $q = 23$, or isomorphic to S_4 with $q = 47$. \square

Theorem 3.13 ([9], Theorem 2.11).

- (a) For $C \leq \mathrm{SL}(2, q)$ cyclic of order $q+1$, we have

$$\mathrm{Aut}(\mathrm{SL}(2, q))_C \cong C_{q+1} \rtimes C_{2e}.$$

- (b) For $q \equiv 3 \pmod{4}$, $q > 7$ and $S \leq \mathrm{SL}(2, q)$ generalized quaternion of order $q+1$, we have

$$\# \mathrm{Aut}(\mathrm{SL}(2, q))_S = e(q+1)$$

and $\mathrm{Aut}(\mathrm{SL}(2, q))_S$ is conjugate to a subgroup of $\mathrm{Aut}(\mathrm{SL}(2, q))_C$ with C cyclic of order $q+1$.

- (c) For $q \in \{23, 47\}$ and $S \leq \mathrm{SL}(2, q)$ of type (b)(iii) in [Proposition 3.12](#) or $q = 7$ and $S \leq \mathrm{SL}(2, 7)$ a quaternion group, we have

$$\mathrm{Aut}(\mathrm{SL}(2, q))_S \cong S_4. \quad \square$$

Corollary 3.14. Let $S \leq \mathrm{SL}(2, q)$ be a subgroup of order $q+1$. Then the stabilizer $\mathrm{Aut}(\mathrm{SL}(2, q))_S$ is solvable. \square

Definition 3.15. Let G be a group. Using the *commutator series* $(G^{(n)})_{n \in \mathbb{N}_0}$, we define the *stable commutator* of G by $G^{(\omega)} := \bigcap_{n \in \mathbb{N}_0} G^{(n)}$.

Theorem 3.16. Let $\mathbb{U}_{S, \mathcal{D}}^{\flat}$ be an $\mathrm{SL}(2, q)$ - \flat -unital of order $q \geq 3$. Then every automorphism of $\mathbb{U}_{S, \mathcal{D}}^{\flat}$ fixes the block $[\infty]$.

Proof. We treat the small cases $q = 3$ and $q = 4$ first: For $q = 3$, there is only one isomorphism type of affine $\mathrm{SL}(2, q)$ -unitals (see [3, Theorem 3.3]), namely the classical affine $\mathrm{SL}(2, q)$ -unital. Hence, its \flat -closure is the Grüning unital and the theorem holds by [5, Lemma 5.5].

For $q = 4$, an exhaustive computer search using GAP showed that there are only two isomorphism types of affine $\mathrm{SL}(2, q)$ -unitals (see [9, Section 6.1.1]) and none of their \flat -closures admit automorphisms which move the block $[\infty]$.

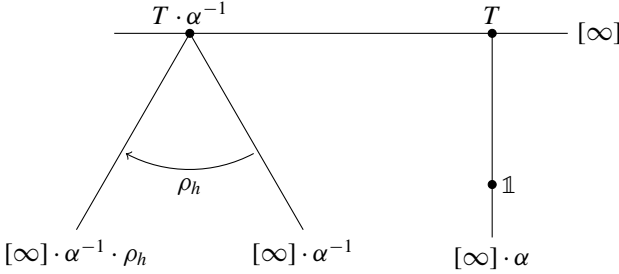
Recall that right multiplication by $\mathrm{SL}(2, q)$ fixes each parallel class of the parallelism \flat and that hence the group $R \leq \mathrm{Aut}(\mathbb{U}_{S, \mathcal{D}}^{\flat})_{[\infty]}$ fixes each point on $[\infty]$. Recall further that we label the points on $[\infty]$ with the Sylow p -subgroups in such

a way that each (affine short) block $T \in \mathfrak{P}$ through $\mathbb{1}$ is incident with the point $T \in [\infty]$.

Assume that there exists an automorphism $\alpha \in \text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ with $[\infty] \cdot \alpha \neq [\infty]$. Since R acts transitively on the affine part of $\mathbb{U}_{S, \mathcal{D}}^b$, assume without restriction $\mathbb{1} \in [\infty] \cdot \alpha$. We distinguish two cases.

Case 1: The block $[\infty] \cdot \alpha$ intersects $[\infty]$, i.e. $[\infty] \cdot \alpha = T$ for $T \leq SL(2, q)$ a Sylow p -subgroup.

Assume first that α does not fix T as a point. Then $[\infty] \cdot \alpha^{-1}$ meets $[\infty]$ in the point $T \cdot \alpha^{-1} \neq T$. Choose $h \in SL(2, q)$ such that $([\infty] \cdot \alpha^{-1})h \neq [\infty] \cdot \alpha^{-1}$. Then the automorphism $\alpha^{-1}\rho_h\alpha$ moves $[\infty]$ and we have $T \cdot (\alpha^{-1}\rho_h\alpha) = T$ as a point, since $T \cdot \alpha^{-1} \in [\infty]$.



We may thus assume without restriction that α fixes T as a point. The group

$$R_T := \{\rho_t \in R \mid t \in T\} \leq R$$

acts regularly on the affine points of the block T and trivially on $[\infty]$. Hence, the group of automorphisms $\alpha R_T \alpha^{-1}$ acts regularly on $[\infty] \setminus \{T\}$ and trivially on $[\infty] \cdot \alpha^{-1}$. In particular, an affine point is fixed and we have

$$q = \#\alpha R_T \alpha^{-1} \mid \#\text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)_{[\infty], \mathbb{1}} \mid \#\text{Aut}(SL(2, q))_S.$$

According to [Theorem 3.13](#), this implies $q \mid 2e(q+1)$ or $q \mid e(q+1)$, since $24 = \#S_4$ is not divided by 7, 23 or 47, respectively. Now $\gcd(q, q+1) = 1$ and $q = p^e > e$ is obtained by an easy induction on e . Hence, $q = p^e \mid 2e$. Since $p^e > e$, this implies $p = 2$ and $2^{e-1} \mid e$. Again, induction shows that $2^{e-1} > e$ if $e \geq 3$. It remains $q \in \{2, 4\}$, but we are only interested in $q \geq 3$ and have already excluded the case $q = 4$.

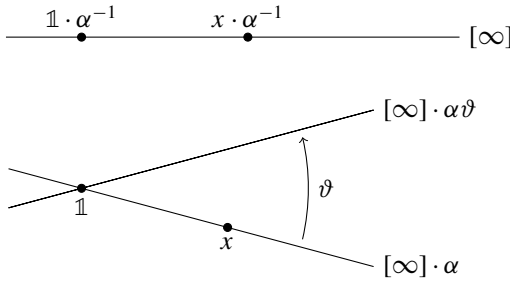
Case 2: The block $[\infty] \cdot \beta$ does not meet $[\infty]$ for any automorphism $\beta \in \text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ with $[\infty] \cdot \beta \neq [\infty]$.

Recall the assumption $\mathbb{1} \in [\infty] \cdot \alpha$. Since $[\infty] \cdot \alpha$ does not intersect $[\infty]$, we know $[\infty] \cdot \alpha \neq T$ for any Sylow p -subgroup $T \leq SL(2, q)$.

We show that no two blocks in the orbit $[\infty] \cdot \text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ may intersect in one point: Let $\vartheta \in \text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ such that $[\infty] \cdot \alpha \neq [\infty] \cdot \alpha\vartheta$ and assume without restriction that the two blocks $[\infty] \cdot \alpha$ and $[\infty] \cdot \alpha\vartheta$ intersect in the point $\mathbb{1}$. Then the automorphism $\alpha\vartheta\alpha^{-1}$ moves the block $[\infty]$. Let $x := \mathbb{1} \cdot \vartheta^{-1} \in [\infty] \cdot \alpha$. Note that, other than indicated in the picture, x need not be different from $\mathbb{1}$. Then $(x \cdot \alpha^{-1}) \in [\infty]$ and

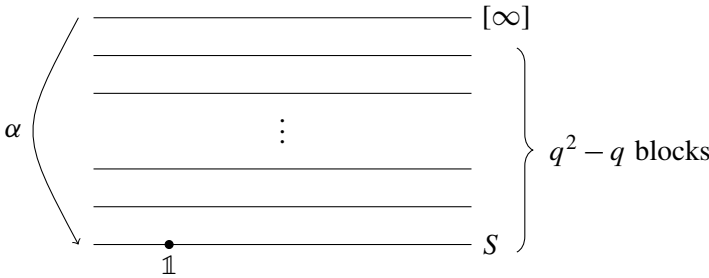
$$(x \cdot \alpha^{-1}) \cdot (\alpha\vartheta\alpha^{-1}) = \mathbb{1} \cdot \alpha^{-1} \in [\infty].$$

Hence, the automorphism $\alpha\vartheta\alpha^{-1}$ moves the block $[\infty]$ to a block intersecting $[\infty]$, a contradiction to the requirement.



For any block $D \in \mathcal{D}$ through $\mathbb{1}$ and $\mathbb{1} \neq d \in D$, the two blocks D and Dd^{-1} intersect in the point $\mathbb{1}$. Since right multiplication by d^{-1} is an automorphism of $\mathbb{U}_{S, \mathcal{D}}^b$, the block $[\infty] \cdot \alpha$ may thus be none of the blocks $D \in \mathcal{D}$. Hence, $[\infty] \cdot \alpha = S$. Since $R \leq \text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ and every affine point is contained in a right coset of S , we have

$$\mathcal{O}_{[\infty]} := [\infty] \cdot \text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b) = \{Sg \mid g \in \text{SL}(2, q)\} \cup \{[\infty]\}.$$



The group of automorphisms R fixes $[\infty]$ and acts transitively on the $q^2 - q$ right cosets of S and hence $\text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ acts 2-transitively on $\mathcal{O}_{[\infty]}$. Since

$$\#\mathcal{O}_{[\infty]} = q^2 - q + 1,$$

the group $\text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ must be a 2-transitive group on $q^2 - q + 1$ elements and contain $R \cong \text{SL}(2, q)$. We use the classification of finite 2-transitive groups to reach a contradiction.

There are several sources for the classification of finite 2-transitive groups, of which the most convenient one seems to be two lists by Cameron, see [1, Tables 7.3 and 7.4]. In the majority of cases, the degree of the action already yields a contradiction, why we only copied the information needed in Tables 1 and 2. In both tables, f is a prime power. In the list of affine 2-transitive groups, G_0 denotes the stabilizer of one point in the 2-transitive action. In the list of almost simple 2-transitive groups, N denotes the minimal normal subgroup of the 2-transitive group G . Indeed, there is one action of degree $9^2 = 3^4$ missing in Cameron's list of affine 2-transitive groups, where $SL(2, 5)$ is a normal subgroup of G_0 , see e. g. Liebeck's proof of Hering's classification of affine 2-transitive groups [7, Appendix 1]. However, we are only interested in the degree in this case and the missing action will not be relevant for our argument.

Let $\Gamma_{[\infty]} := \text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)_{[\infty]}$ denote the stabilizer of $[\infty] \in \mathcal{O}_{[\infty]}$ in our required 2-transitive action of $\text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ on $\mathcal{O}_{[\infty]}$. Then

$$SL(2, q) \cong R \leq \Gamma_{[\infty]} \leq \mathfrak{A}_S \times R \cong \text{Aut}(SL(2, q))_S \times SL(2, q).$$

In particular, we know that the stable commutator $\Gamma_{[\infty]}^{(\omega)}$ equals $SL(2, q)$ for $q \geq 4$, since $\text{Aut}(SL(2, q))_S$ is solvable for each choice of S , see Corollary 3.14. In the first affine 2-transitive action in Table 1, the stable commutator $G_0^{(\omega)}$ equals the stable commutator of $SL(d, f)$, which is trivial or $SL(d, f)$. Since all isomorphisms between finite (projective) special linear groups are known (see e. g. [6, Satz 6.14]),

degree	G_0	condition
f^d	$SL(d, f) \leq G_0 \leq \Gamma L(d, f)$	$f = 5^2, 7^2, 11^2, 23^2, 3^4, 11^2, 19^2, 29^2, 59^2, 2^4, 2^6, 3^6$
f^{2d}		
f^6		
f		

Table 1. Degrees of affine 2-transitive groups.

degree	N	condition
n	A_n	$n \geq 5$
$(f^d - 1)/(f - 1)$	$PSL(d, f)$	$d \geq 2, (d, f) \neq (2, 2), (2, 3)$
$2^{2d-1} \pm 2^{d-1}$		$d \geq 3$
$f^3 + 1$		$f = 11, 12, 15, 22, 23, 24, 28, 176, 276$
$f^2 + 1$		
f		

Table 2. Degrees of almost simple 2-transitive groups.

we conclude $d = 2$ and $f = q$. But this is not possible because of our required degree $q^2 - q + 1 \neq q^2$ and we may thus exclude the first affine 2-transitive action.

Every other affine 2-transitive action is excluded by the observation that their degree is always a square but our degree $q^2 - q + 1$ is never a square, since

$$(q - 1)^2 < q^2 - q + 1 < q^2.$$

Now we look at the list of almost simple 2-transitive groups. The point stabilizer of the action of A_n is A_{n-1} , which is far bigger than $\Gamma_{[\infty]}$ for $n = q^2 - q + 1$.

The second entry in [Table 2](#) is the action on the points or hyperplanes of a projective space. For $d \geq 3$ and $(d - 1, f) \notin \{(2, 2), (2, 3)\}$, the stable commutator of the point stabilizer of this action is the special affine group

$$\text{ASL}(d - 1, f) = \text{SL}(d - 1, f) \rtimes \mathbb{F}_f^{d-1}.$$

Hence, $\text{ASL}(d - 1, f) \cong \text{SL}(2, q)$ (recall $q \geq 4$). But $\text{ASL}(d - 1, f)$ contains the normal subgroup \mathbb{F}_f^{d-1} of order $f^{d-1} > 4$, while the highest possible order of a normal subgroup of $\text{SL}(2, q)$ is 2 for $q \geq 4$. For $(d - 1, f) \in \{(2, 2), (2, 3)\}$ the degree of the action would be 7 resp. 13 and hence $q = 3$ or $q = 4$, which we already excluded in the beginning. It remains the case $d = 2$. Then the degree is $f + 1$ and hence $f = q^2 - q = q(q - 1)$. But $\text{gcd}(q, q - 1) = 1$ shows that the product $q(q - 1)$ is not a prime power. Hence we may exclude the second entry in the list of almost simple 2-transitive actions.

For the remaining cases, considering the degree will suffice. The degrees $2^{2d-1} \pm 2^{d-1}$ ($d \geq 3$) and 12, 22, 24, 28, 176, 276 are excluded by the observation that $q^2 - q + 1$ is odd. Then we just showed that $q^2 - q$ cannot be a prime power f^d . Finally, it is easy to see that $q^2 - q = q(q - 1) \notin \{10, 14, 22\}$ for any prime power q .

Thus, $\text{Aut}(\mathbb{U}_{S, \mathcal{D}}^b)$ does not act 2-transitively on $\mathcal{O}_{[\infty]}$ and our proof is complete. \square

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VERENA MÖHLER:

verena.moehler@partner.kit.edu

Karlsruhe, Germany

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